One Dimensional Conductive Cooling of a Sill using Difference Equations

$$\frac{d}{dt}\cdot f(x,t)=\kappa\cdot\frac{d^2}{dx^2}\cdot f(x,t)$$

is the 1D, time-dependent heat conduction equation; κ is diffusivity.

The first step in setting up a difference equation to numerically solve this equation is to express the derivatives as discrete expressions. By definition,

$$\frac{d}{dx}f(x,t)=\lim_{\Delta x \to 0}\frac{f(x+\Delta x,t)-f(x,t)}{\Delta x}$$

Which we approximate by: $$\frac{d}{dx}f(x,t)=\frac{f(x+\Delta x,t)-f(x,t)}{\Delta x}$$

This is usually the fastest way to calculate a numerical derivative, is sometimes an appropriate way, but is rarely the best way. Most numerical analysis texts (e.g. Press et al., Numerical Recipes) include a long development of the topic. However, this approach will suffice for the problem at hand. For the second derivative, just repeat the procedure

$$\frac{d}{dx}\left(\frac{d}{dx}f(x,t)\right)=\frac{\frac{f(x+\Delta x,t)-f(x,t)}{\Delta x}-\frac{f(x,t)-f(x-\Delta x,t)}{\Delta x}}{\Delta x}$$

$$\frac{d}{dx}\left(\frac{d}{dx}f(x,t)\right)=\frac{f(x+\Delta x,t)-f(x,t)}{\Delta x^2}-\frac{f(x,t)-f(x-\Delta x,t)}{\Delta x^2}$$

$$\frac{d}{dx}\left(\frac{d}{dx}f(x,t)\right)=\frac{f(x+\Delta x,t)-2\cdot f(x,t)+f(x-\Delta x,t)}{\Delta x^2}$$

$$\frac{d^2}{dx^2}f(x,t)=\frac{f(x+\Delta x,t)-2\cdot f(x,t)+f(x-\Delta x,t)}{\Delta x^2}$$

Similarly:

$$\frac{d}{dt}f(x,t)=\frac{f(x,t+\Delta t)-f(x,t)}{\Delta t}$$

Thus we can write the one dimensional, time dependent heat conduction equation as:

$$\frac{f(x,t+\Delta t)-f(x,t)}{\Delta t}=\kappa\cdot\left(\frac{f(x+\Delta x,t)-2\cdot f(x,t)+f(x-\Delta x,t)}{\Delta x^2}\right)$$

What we really want to know is how the system progresses with time. Thus we solve the previous equation for f(x,t+Δt).

$$f(x,t+\Delta t) = \frac{\kappa \cdot \Delta t}{\Delta x^2} \cdot (f(x+\Delta x,t) - 2 \cdot f(x,t) + f(x-\Delta x,t)) + f(x,t)$$

The previous equation is our desired expression, the 1D time dependent heat conduction equation expressed as a finite difference equation. We now need to cast it in a form suitable for Mathcad. The density, specific heat, and thermal conductivity are:

$\rho := 2700 \cdot \frac{kg}{m^3}$ $\quad Cp := 750 \cdot \frac{J}{kg \cdot C}$ $\quad k := 2 \frac{W}{m \cdot C}$

$\kappa := \frac{k}{\rho \cdot Cp}$ $\quad$ κ (kappa) is diffusivity which is m^2/s (a watt is a Joule/sec)

$\kappa = 9.877 \cdot 10^{-7} \cdot m^2 \cdot s^{-1}$

Next, change the origin to zero, and set up the time and distance steps:

$ORIGIN = -1$ ... set this in the Math menus so we can start values at zero and increment by 5.

$t := 0, 1 .. 1000$ ... will increment time steps for the iterative solution.

Now, we'll set Δt and Δx and combine them with the thermal diffusivity, κ.

$\Delta x := 25\ m$ $\quad \Delta t := 60 \cdot 60 \cdot 24 \cdot 365 \cdot 5\ s$ $\quad \frac{\Delta t}{60 \cdot 60 \cdot 24 \cdot 365} = 5 \cdot s$ years in seconds

$xmax := 1150 \cdot m$ $\quad x := 0, \Delta x .. xmax$ . . . x is the distance coordinate in meters

Next, change the origin to zero, and set up the time and distance steps:

$ORIGIN := -25$ ... set equal to -Δx

$SillTop := 800 \cdot m$ $\quad SillThick := 6 \cdot \Delta x$ $\quad SillThick = 150 \cdot m$

$Sill := SillTop, SillTop + \Delta x .. SillTop + SillThick$ ... the x depth range where the sill is.

$\kappa \cdot \frac{\Delta t}{\Delta x^2} = 0.249$ ... κ is diffusivity, dimensionless, and needs to be << 0.50 for stability

Now the initial conditions at time t = 0.

$grad := .03 \cdot \frac{C}{m}$ ... use grad for the initial temperature gradient

$f_{0,x} := grad \cdot x$ ... set the temperature equal to the distance.

$f_{0,0} := 0 \qquad f_{0,-\Delta x} := 0$ ... starting temperature at the origin is zero

$f_{t,xmax+\Delta x} := f_{0,xmax} + xmax \cdot grad$ ... temperature at end is held at this temperature

$f_{0,Sill} := 1000 \cdot C + f_{0,Sill}$ ... the dike gets emplaced into the temperature gradient. This, and the graph below, shows that the two temperature regimes are simply added together.

$$f_{t+1,x} := \kappa \cdot \frac{\Delta t}{\Delta x^2} \cdot \left(f_{t,x+\Delta x} - 2 \cdot f_{t,x} + f_{t,x-\Delta x} \right) + f_{t,x}$$