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TRACING STUDENTS' MODELING PROCESSES IN SCHOOL

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Abstract

In this study we report on an analysis of the mathematization processes of one 6th and one 8th grade group, with emphasis on the similarities and differences between the two groups in solving a modeling problem. Results provide evidence that all students developed the necessary mathematical constructs and processes to actively solve the problem through meaningful problem solving. Eighth graders were involved in a higher level of understanding the problem presented in the activity, employed more sophisticated mathematical concepts and operations, better validated and communicated their results and reached more efficient models. Finally, a reflection on the differences in the diversity and sophistication of the constructed models and mathematization processes between the two groups raises issues regarding the design and implementation of modeling activities in elementary and lower secondary school level.

INTRODUCTION

A number of professional organizations (AAAS, 1998; NCTM, 2000) address the need for a change in the school mathematics. They also propose reforms in mathematics education to fulfill the economy and work force's demands for school graduates that are able to possess flexible and creative mathematical problem solving abilities and to effectively use technological tools in working collaboratively in demanding projects (NCTM, 2000). An effective medium for achieving the above demand for a change in school mathematics is modeling; the study of mathematical concepts and operations within real world contexts and the construction of models in exploring and understanding real complex problem situations (English, 2006).

Although mathematical modeling has been reserved for the upper secondary and tertiary education levels (Greer, Verschaffel, & Mukhopadhyay, 2007), recent research shows that students in primary and lower secondary school level can effectively work with modeling activities; constructing appropriate and effective models for solving complex real world problems and effectively using where available technological tools (e.g., English, 2006; English & Watters, 2005; Mousoulides, 2007). The present study aims to further contribute to our understandings of students' modeling processes and mathematical developments in elementary and lower secondary school, by examining similarities and differences between 6th and 8th grade students' modeling processes and mathematical developments. This identification is expected to further contribute to the appropriate introduction of modeling in elementary school mathematics.

THEORETICAL FRAMEWORK

Mathematical modeling has been considered as an effective medium to prepare students to deal with unfamiliar situations by thinking flexibly and creatively and to solve real world problems (Lesh & Doerr, 2003; English, 2006). Although the National Council of Teachers of Mathematics (NCTM, 2000) calls for purposeful activities along with skillful questioning to promote the understanding of relationships among mathematical ideas, this recommendation can be pushed further and modeling activities can be used as a way to cultivate critical thinking and critical literacy (Mousoulides, 2007). Related research indicated that student work with modeling activities assisted students to build on their existing understandings and to be successfully engaged in thought-provoking, multifaceted complex problems (Lesh & Doerr, 2003; English, 2006). Modeling activities set within authentic contexts, allow for students' multiple interpretations and approaches, promoting intrinsic motivation and self regulation. Students are also engaged in important mathematical processes such as describing, analyzing, constructing, and reasoning as they mathematize objects, relations, patterns, or rules (Lesh & Doerr, 2003).

Research in the field of mathematical modeling listed six design principles for developing modeling activities that are based on the work of teachers and researchers and that have subsequently been refined by Lesh and his colleagues (2000). The Model Construction Principle ensures that the solution to the activity requires the construction of an explicit description, explanation, procedure, or justified prediction for a given mathematically significant situation. The Reality Principle requires that students can interpret the activity meaningfully from their different levels of mathematical ability and prior knowledge. The Self-Assessment Principle ensures the inclusion of criteria that the students themselves can identify and use to test and revise their current ways of thinking. Specifically, the modeling activity should include information that students can use for assessing the usefulness of their alternative solutions, for judging when and how their solutions need to be improved, and for knowing when they are finished. The Model Documentation Principle ensures that while completing the modeling activity, the students are required to create some form of documentation that will reveal explicitly how they are thinking about the problem situation. The fifth principle is the Construct Share-Ability and Re-Usability Principle, which requires students to produce share-able and re-usable solutions, that can be used by others beyond the immediate situation. The Effective Prototype Principle ensures that the modeling activity is as simple as possible yet still mathematically significant. The goal is for students to develop solutions that will provide useful prototypes for interpreting other similar situations.

During the last years, an increasing number of researchers have focused their research efforts on mathematical modeling, especially at the school level (e.g., English, 2006, 2003). It is a necessity to implement worthwhile modeling experiences in the elementary and middle school years if teachers are to make mathematical modeling a successful way of problem solving for students (English, 2006). Modeling activities differ from traditional approaches to the teaching of elementary mathematics for a number of reasons. First, the mathematical concepts and operations that are needed to successfully solve the problems appear in the modeling activities go beyond what is taught and how in traditional mathematics classrooms. Additionally, mathematical concepts are presented with connections to real world situations and students have the opportunities to further elaborate on the related concepts and build on their prior understandings (English, 2006).

In modeling problems students are offered rich learning opportunities. The problems presented are not carefully mathematized for the students, and therefore students have to unmask the mathematics by mapping the problem information in such a way as to produce an answer using familiar quantities and basic operations. Students mathematize the problems in ways that are meaningful to them, and this process can result in improving competencies in using mathematics to solve problems beyond the classroom (English, 2003; Mousoulides, 2007). A second characteristic of modeling activities for elementary school students is that they encourage the development of generalizable solutions. These solutions (models) focus on the structural characteristics of the problems (systems) that they are referring to, and are expressed using a variety of representational media, including written symbols, diagrams or graphs. The latter is central in mathematics learning for elementary school students (Lesh & Doerr, 2003). In the activity presented in this study, student models for rates, for example, might include working hours and money collected, operations such as multiplication and division, and relationships between working hours and money collected that stand for productivity rate. In addition, models incorporate a number of external representations (e.g., graphs, tables). In constructing models, students identify, select and collect relevant data, interpret the solution in context, describe situations using a variety of representation media and document and communicate their solutions (Lesh & Doerr, 2003).

Students in the elementary and lower secondary school level can be benefited from working with authentic modeling problems (English & Watters, 2005; English, 2006; Mousoulides, 2007). The use of modeling activities encourage students to develop important mathematical ideas and processes that students normally would not meet in the traditional school curriculum. The mathematical ideas are embedded within meaningful real-world contexts and are elicited by the students as they work the problem. Furthermore, students can access these mathematical ideas at varying levels of sophistication (English, 2006). Student work in modeling activities facilitates student development of generalizable conceptual systems, as students move beyond just thinking about their models to thinking with them for solving real problems (Lesh & Doerr, 2003). English (2006) also reported that there was considerable evidence that students' mathematical ideas, mathematical language and fluency in using tables and data were improved after they worked in a sequence of modeling activities.

THE PRESENT STUDY

The Purpose of the Study

The purpose of the present study was twofold; first to examine students' modeling and mathematization processes as they worked on a mathematical modeling problem and second to investigate similarities and differences between 6th and 8th grade (11 and 13 year olds) students' modeling and mathematization processes. The problem addressed in this study, "University Cafeteria", required students to construct models for selecting the best among a number of vendors. Specifically, students were given two tables presenting the hours worked and money collected for nine vendors in a university cafeteria (see Figure 1). Based on the provided data, students needed to construct models for selecting three full time and three part time vendors for next year.

	Hours Worked Last Year								
	Autumn Semester			**Spring Semester**			**Summer Period**		
	Busy	**Steady**	**Slow**	**Busy**	**Steady**	**Slow**	**Busy**	**Steady**	**Slow**
Maria	12.5	15	9	10	14	17.5	12.5	33.5	35
Kim	5.5	22	15.5	53.5	40	15.5	50	14	23.5
Terry	12	17	14.5	20	25	21.5	19.5	20.5	24.5
Jose	19.5	30.5	34	20	31	14	22	19.5	36
Chad	19.5	26	0	36	15.5	27	30	24	4.5
Cheri	13	4.5	12	33.5	37.5	6.5	16	24	16.5
Robin	26.5	43.5	27	67	26	3	41.5	58	5.5
Tony	7.5	16	25	16	45.5	51	7.5	42	84
Willy	0	3	4.5	38	17.5	39	37	22	12

	Money Collected Last Year								
	Autumn Semester			**Spring Semester**			**Summer Period**		
	Busy	**Steady**	**Slow**	**Busy**	**Steady**	**Slow**	**Busy**	**Steady**	**Slow**
Maria	690	780	452	699	758	835	788	1732	1462
Kim	474	874	406	4612	2032	477	4500	834	712
Terry	1047	667	284	1389	804	450	1062	806	491
Jose	1263	1188	765	1584	1668	449	1822	1276	1358
Chad	1264	1172	0	2477	681	548	1923	1130	89
Cheri	1115	278	574	2972	2399	231	1322	1594	577
Robin	2253	1702	610	4470	993	75	2754	2327	87
Tony	550	903	928	1296	2360	2610	615	2184	2518
Willy	0	125	64	3073	767	768	3005	1253	253

Figure 1. The number of hours worked and money collected by each vendor.

Participants, Modeling Activity, and Procedure

One intact class of 6th grade students (12 females and 7 males) and one intact class of 8th graders (10 females and 8 males) from two urban schools in Cyprus participated in the "University Cafeteria" modeling activity. The activity is a modified version of the "Summer Jobs" activity (see Lesh & Doerr, 2003). Students in both classes had prior experience in working with modeling activities; they participated in a two year project for implementing modeling activities in elementary and lower secondary schools (see Mousoulides, 2007). Specifically, prior working in the activity presented in this study, students worked in four modeling activities.

Students worked for four 40 minute sessions to find a solution for the problem presented in the activity. The purpose of the activity was to provide opportunities for students to organize and explore data, to use proportional reasoning and to develop appropriate models for solving the problem. Additionally, the activity provided a setting for students to work with the notions of ranking, selecting, aggregating ranked quantities and weighting ranks. During the first part of the activity, which lasted around 20 minutes, students were presented with a newspaper article about the different factors employers take into consideration when hiring vendors, followed by readiness questions. Following, during the modeling stage of the activity (80-90 minutes) students worked in groups of three, using spreadsheet software, to solve the modeling problem. In the last part of the activity, students presented their models in whole class presentations for reviewing and discussing with their peers. Finally, a whole class discussion focused on the key mathematical ideas and processes that were developed during the activity. For the purposes of the present study the results from one representative group of students in each grade level during the modeling stage of the modeling activity are presented in this study.

Data Sources and Analysis

The data for this study were collected through (a) audiotapes of students' work in their groups, (b) spreadsheet files from students' work, (c) students' worksheets, detailing the processes used in developing their models, and (d) researchers' field notes. The analysis of the data mainly focuses on the results of one group of students in each grade, as they worked on the modeling stage of the activity. The selected groups were representative of the two classes, in a way that other groups in their classes reported similar work.

The analysis of the data was completed in the following steps. First, the transcript was reviewed several times to identify the ways in which students interpreted the problem, their approaches to use data from the two tables, and their mathematization processes in linking data from the tables. The transcript was also reviewed to identify how students interacted in their group, and how discussion within the group resulted to a final model. Finally, all students' worksheets, spreadsheet files, and their final written letters were analyzed to identify and compare the mathematization processes used in their model development.

RESULTS

The results of the study are presented in terms of the mathematization processes presented by each group of students. Specifically, the modeling processes, appeared in students' work, are presented for each group with regard to the steps of the modeling procedure (Description, Manipulation, Prediction of the problem, and Solution Verification). The respective students' mathematical developments are presented in three cycles, with regard of increased sophistication of mathematical thinking.

Modeling Processes

Sixth graders failed to fully understand the problem; they understood the core question of the problem but they did not succeed in connecting the core question with the provided data. As a result, they only focused on isolated parts of the data from the money collected table. Students' initial models were inadequate to solve the problem and only researchers' comments helped students to overcome these difficulties. On the contrary, eighth graders made the necessary connections and almost immediately merged data from both tables, and identified patterns and relations. However, their first attempts focused on specific vendors and they commented on either the amount of money collected or either the hours each vendor. It was apparent that 8th grade students identified the necessary variables and relationships to describe and understand the problem. However, they failed to handle and relate these understandings with the core question of the problem.

During problem manipulation, both groups of students presented a number of interesting models. A number of differences can be tracked between 6th and 8th grade students' models. Sixth graders' initial model was based on ranking the nine vendors in each column and then finding a total ranking for autumn semester. Students proceeded on a second model, which was based on calculating the total amount of money each vendor earned, when they realized that their first model was not efficient enough. However, even this new second model was not appropriate since it was not based on data from both tables. Students' next (last) model was resulted after a discussion with the researchers; this "performance rate" model could answer (to some extent) the core question of the problem. Eighth graders' work resulted in better and more sophisticated models, which could answer the core question of the problem. Specifically, students easily reached a model based on the "performance rate" for each vendor and used this model in selecting the best six vendors. This model was based on calculating the total amount of money collected by each vendor and divided by the total number of hours each vendor worked. However, 8th graders' first model did not take into consideration the semester and/or time period factors. Eighth graders further improved their first model by firstly employing the semester dimension and second the time period dimension. These two final more sophisticated models extensively used the whole data set, including constraints, parameters and patterns in the data.

The fact that 6th graders did not interpret their results in the context of the real problem was one of the reasons that their solutions were not successful enough. Students failed to examine the appropriateness of their model and to discuss issues related to model's interpretation. In the case that two vendors were quite closed in terms of money collected, for example, there were long debates on deciding which vendor to select. However, 8th graders did not employ semester and/or time period factors. A possible reason was that students might lack the necessary mathematical concepts to better mathematize the real problem and therefore to construct a refined model. Eighth graders made significant efforts not only to interpret their model in the context of the real problem but also to examine different models and to make the selection based on this interpretation. Students' first interpretations questioned the appropriateness of their model, since that model could not help them in selecting the best vendors. A second dimension of models' interpretation and examination was presented in students' final discussion. Students' work resulted in two different but mathematically correct models. One model was based on the different semesters and the second on the different time periods. At that time, students asserted that being competent in different time periods was more important than being competent in different semesters and they concluded in adopting the time periods model.

A number of differences related to the verification of the solutions appeared between 6th and 8th grade students' work. Sixth graders did not actually verify their solution, but they only compared their last model to previous ones and made comments. This was a major disadvantage of their work and blocked their efforts to further improve their model. On the contrary, 8th graders not only reached two mathematically correct models, but they also based their final decision on the context of the real problem. Students finally chose the time periods based model and not the semester one, while documenting and supporting their decision. In terms of documentation and communication, students expressed their ideas and solutions not only verbally, but using a variety of representational media, including different graphs (bar chart, line graphs) and sketches. Students' efforts to convince the imaginary client (cafeteria manager) about the correctness of their solution encouraged students to reflect on their models. In their letters, students documented their results about the specific problem and tried to provide solutions for structurally similar problems. In comparing students' work in terms of communicating their results, three differences can be extracted. Eighth grade students used a variety of representations in documenting and explaining their results in their letters. The second difference was again located in students' letters to the cafeteria's manager. Eighth graders' letters were presented in details, and were based on students' previous approaches and models. The third difference relied on the discussions students had in their groups. Eighth grade students extensively discussed most of the issues that arose during their investigations. On the contrary, 6th graders' discussions were of less importance and in many times students just expressed their ideas, without trying to elaborate on their peers' ideas.

Mathematical Developments

Students' mathematical developments are summarised in terms of cycles of increased sophistication of mathematical thinking, with each cycle representing a shift in thinking. The analysis of students' mathematical developments is presented in the following order. First, students' efforts were limited to focusing on subsets of information. Second, students started using mathematical operations (e.g., finding ratios) and third students' work was based on more sophisticated mathematical ideas, such as identifying trends and relationships among the data.

Sixth graders commenced the activity by scanning the money table to find vendors who scored highly in one or more columns (i.e., money collected in busy, steady or slow time periods, in autumn or spring semester etc). Only limited mathematical thinking was displayed in students' unsystematic work. This was also evident in students' comments: "Jose and Chad both worked 19.5 hours. Wow, Robin worked more. She worked 26.5 hours. She is first". Students decided to use data from both tables only when they failed to find a solution. However, students' approach still remained unsystematic and isolated as they did not manage to "merge" data from both tables. As a result, students still used descriptive comments: "Robin worked more hours and earned more money than any other vendor". The inexistence of any systematic approach resulted in contradictions, which generated the need to further mathematize their approach. The group began to use two mathematical operations to aggregate the data for each vendor, namely: (a) simply totalling the amount of money each vendor earned and how many hours each vendor worked, and (b) finding the average for each category (money, hours) and classifying vendors above and below average.

Similarly to 6th grade group's work, 8th grade students commenced the activity by scanning the two tables to find the vendors who scored highly in one or more columns. This initial approach can be characterized as unsystematic. Students, for example, made comments like "Look! Robin worked more hours than anybody else" or "Tony might was on vacations (when Tony worked only for three hours)". Students did not pay attention to the column headings. As a consequence, when they started mathematizing the problem, their first model was based on finding the total number of hours and money for each vendor. In contrast to 6th graders, 8th graders immediately realised that they had to use data from both tables. However, students' approach still remained unsystematic and limited as they only commented on specific vendors, without making any connections between the two tables. These results demonstrated the inefficiency of their approaches and encouraged students to mathematize their work. Students used mathematical operations to aggregate the data for each vendor, namely finding for each vendor the total amount of money earned and the total number of hours worked. They finally used the two rankings for selecting the six vendors.

The second cycle of students' mathematical developments can be characterized by the mathematical operations students used in constructing their models. Sixth graders acknowledged the weakness of their approach and suggested finding for each vendor the total amount of money collected in autumn semester. They justified their decision by explaining that: "It's difficult to find the best in each column. Maria is sometimes amongst the best ones and in other columns she is amongst the worst vendors". To resolve the issue and to better classify vendors, students decided to find the average and then classify vendors in two categories; above and below average. However, students only used data from the hours worked table. A second dimension of 6th graders' work was the misinterpretation of the hours worked table. Specifically, among two vendors who collected the same amount of money students chose the vendor who worked more hours. When 8th graders faced the need to rank the nine vendors, they decided to total the number of hours for each vendor and the total amount of money each vendor earned during the three semesters. They justified their decision by explaining that: "We can find one ranking for the hours worked and one ranking for the money collected. We can then use these rankings to select the three vendors that will work full time and the three that will work part time". However, the above model was not efficient for answering the core question of the problem, since the two rankings were contradictory. Terry, for example, was fifth in the first and ninth in the second ranking.

During the third cycle of mathematical developments, students' work can be characterized by the identification of trends and relationships in the data. Sixth graders realised, after discussing with the researcher, that their average based model was not appropriate enough and they focused their efforts on finding a relationship between money collected and hours worked for each vendor. The acknowledgement of these requirements led students to progress to the notion of rate. However, the notion or rate was partially employed, since students did not take into consideration the different time periods (busy, steady and slow). Eighth graders decided to proceed in finding the money per hour ratio for resolving the conflicts they faced and for classifying the nine vendors. At the same time, substantial discussion and argumentation took place when students tried to find a way to merge data from both tables. Students progressed to looking for a relation between money collected and hours worked. Students quite easily identified the relationship between money and hours and they constructed a model for calculating the money per hour ratio. Part of this discussion focused on "vendor's productivity". Specifically, students agreed upon selecting the most productive vendors and they defined productivity as the amount of money each vendor earned in one hour. The acknowledgement of this new parameter directed students' "performance rate" model.

Eighth grade students were concerned that using the "performance rate" model resulted in a totally different ranking. Although it was apparent that students were satisfied with the solution, they questioned the appropriateness of their model. Consequently, students refined their last model by finding money per hour ratio for each semester and for each time period. These two new rankings were completely different than the first one and there were also differences between them. Students decided to adopt this time period based model, since "it is reasonable to base our selection on the different time periods instead of the different semesters. It is important that one vendor is good in all time periods". However, students failed to use these assumptions in further refining their model (e.g., weighting time periods).

DISCUSSION

A significant finding of the present study, which is in line with other studies (e.g., English, 2003, 2006), is that elementary and lower secondary school students are able to successfully work with mathematical modeling activities when presented as meaningful, real world problems. The framework, within the "University Cafeteria" activity, helped students to realize and to get familiar with the problem situation and thus enhanced their understandings. Students in both groups progressed from focusing on subsets of information which resulted in not efficient models to applying the appropriate mathematical concepts and operations that helped them finding an appropriate mathematical model based on the "performance rate". Their models were reusable, shareable and could serve to construct more sophisticated models for solving more demanding problems (Doerr & English, 2003). The results of the study provide evidence that students in both groups developed the necessary mathematical constructs and modeling processes to actively engage and solve the problem through meaningful problem solving. Students in both groups effectively used the spreadsheet software to represent their data in graphical and symbolic form and to find ratios for ranking the different vendors.

Among the differences between the two groups, 8^{th} grade students more easily identified the necessary variables and relationships to describe and understand the problem, taking into consideration data from both tables. Additionally, 8^{th} graders presented more refined and sophisticated solutions and successfully solved the problem by employing two "performance rate" models, based on the different semesters and time periods. A third difference between 6th and 8th graders was from the perspective of communication and assessment. Although students in both groups adequately communicated their ideas and solutions, only 8th graders undertook constructive assessment by listening to and reflecting on their peers' suggestions and models. On the contrary, 6th graders mainly presented their personal ideas in their discussions and not reflected on others' ideas and suggestions.

Although it is difficult to explain the differences between the two groups of students, given the plurality of possible reasons, it is nevertheless important to examine how these differences might influence the design and implementation of modeling activities in elementary and lower secondary school level. These differences could account in part for the variation in diversity and sophistication of the models the students created. The design of modeling activities should take into consideration students' mathematical developments and therefore do not constrain students' efforts in solving such problems. Additionally, what was evident from the present study was the importance of the researcher's – teacher's role in overcoming the difficulties that arose in, mainly, 6th graders' work. In concluding, the findings of the study shows that although modeling activities can be successfully implemented in elementary and lower secondary school and that can improve students' mathematical understandings, further research is needed towards this direction.

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