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# Mathematical Objects and the Evolution of Rigor

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[In memory of Guillermina Waldegg]

ABSTRACT: *In this paper we discuss the origins and the evolution of rigor in mathematics in relation to the creation of mathematical objects. We provide examples of key moments in the development of mathematics that support our thesis that the nature of mathematical objects is co-substantia1 with the operational inventions that accompany them and that determine the normativity to which they are subjected.*

## THE HISTORICAL DEVELOPMENT AND NORMATIVITY OF MATHEMATICAL OBJECTS

Mathematical experience has brought us all, at some point in our intellectual development, up against the belief that we are facing "true" knowledge. This feeling is reinforced as we come into contact with mathematical proofs, almost always during elementary geometry c1asses. Much later - if at all - we encounter a perturbing question: Of what truth does mathematics speak? The concern arises because our traditional image of "truth" refers to an apparent correspondence between intellectual constructs and the outside world that we generally call "objective reality". It is then inevitable that we ask what reality our mathematical statements conform to, what objects are predicated, and with what criteria the truthfulness of those predicates is judged.

To discuss mathematical objects and rigor, we must simultaneously consider the historical point of view and the normative point of view. From these perspectives, we will define two methodological principles that will be exemplified in the course of this paper. The first principle maintains that the mathematical object cannot be separated from its normativity. By normativity we mean the criteria of validity in relation both to the actions that lead to the construction of the mathematical object and to the actions and operations that can be validly performed on the constructed object. We hold that the object, the actions involved in its construction, and the operations that can be validly performed on it cannot be disassociated from each other. Throughout history, as we shall argue in this paper, attempts to separate objects from their normativity have plunged mathematics into crisis situations. The second principle denies the possibility of explaining a certain level of organization of knowledge by reducing it to a lower level. Based on the idea that the creation of a mathematical object is a historical process with a strong epistemological component, we claim that explanations of scientific processes are possible only if a lower level of conceptual development is explained from a higher level.

**OUR FIRST EXAMPLE**

The mathematical object cannot be disassociated from the forms of operational intervention that are possible with it and on it. In certain highly important instances, exploration by means of operational fields has been so profound as to modify the very course of mathematics. Perhaps one such occasion was what is known as the "crisis of the incommensurable" in the Pythagorean School[1]. Laboring under a view that saw the world organized on the basis of the integer numbers and their ratios, the Pythagoreans themselves erected a formidable obstacle. They explored the operational fields of their concept of number and took them to the very limits, only to find there an unexpected result: the incommensurability of two segments. This fact shattered their concept of mathematics and, since the two were inseparable, their very concept of the world as well. The Greek philosopher Proclus wrote that the Pythagorean who made this discovery of incommensurability was put to death via shipwreck.

The Pythagoreans therefore decided not to accept the consequences of their reasoning, thus breaking the permanent ties that exist between the object and its operational fields. In other words, in not accepting the consequences of the operations because they revealed a property of number that did not concur with their preconceptions, the Pythagoreans were forced to modify their notion of number, thus excluding the numeric representation of continuous magnitudes (endnote iii). This led to a separation between arithmetic and geometry, between the domain of the discrete and the domain of the continuous. The repercussions of this decision were felt during a long period in the

1 Some math historians claim that there is no evidence per se of any crisis in the Pythagorean School of the kind we describe, apart from an anecdotal remark from Proclus one thousand years later. Even if one believes that the crisis was a sort of a myth, the mathematical issue of somehow resolving the problem of incommensurability existed and led to the separation of arithmetic and geometry. We are not trying to re-construct ancient Greek mathematics, but are simply exploring the consequences of breaking ties between mathematical objects and their operational fields.

history of mathematics that ended only in the 16th century. We will return to this point later. This example illustrates the fact that even when operational interventions are accepted as a way to construct the object, there are still critical situations that arise, thresholds that mathematicians refuse to cross on account of the ontological beliefs they hold prior to the operational exercise.

### THE ORGANIZATION OF MATHEMATICS

From an epistemological point of view, the historical development of mathematics coincides with an increase in its levels of organization. Euclidean mathematics, for example, corresponds to an earlier organizational level than Hilbert's; this does not mean that Hilbertian[2] mathematics was latent within Euclid, although certain similarities, as well as certain divergences, between these two fields of rationality must be acknowledged. Obviously, they correspond to two radically different styles of mathematical practice and conceptualization (and therein lies the divergence); nevertheless, in spontaneous attempts to explain the relations between the two theories, there is a tendency to speak of the higher level by reducing its level of complexity and taking its elements down to the lower level. Conversely, the lower level finds its explanation when the higher one assimilates it. In more local terms, this need to explain phenomena through resorting to a higher level of organization can be seen when we study a proof that for reasons of complexity or length has been divided into a series of auxiliary propositions. A good example is the proof of the theorem which states that the product of two compact spaces is compact. The proof relies on a series of prior lemmas, and culminates in a terse sentence which states the theorem is proved by invoking the previous lemma. Although we can understand each of the parts, we are unable to shake off a feeling of opaqueness, of failing to understand the theorem in question. This is because the global meaning has been lost by subdividing the proof, and this imposes the need to interconnect the parts in order to recover the sense of the whole. The evolution of the dialectic between the object and its normativity is thus explained as part of the global organization of mathematics.

### OUR SECOND EXAMPLE: LOCAL DOMAINS OF INTELLIGIBILITY

The global organization mentioned above refers to mathematics at a moment in its development, when we "stop" the constructive process. Mathematics undergoing construction obeys an organizational dynamic that is at first local in nature. Initially, objects and situations do not appear clearly drawn: the object appears within a conceptual network, equipped with a provisional operational field that serves to begin its exploration. The history of mathematics shows us that during the evolution of a discipline, conceptual nuclei are formed and mathematical activity progresses around

[2] By Hilbertian mathematics, we mean Hilbert's axiomatic re-organization of Euclidean geometry.

them. We shall call these nuclei local domains of intelligibility. Consider an example taken from differential calculus (endnote ii). During the 17th century, maxima and minima problems were identified with problems involving plotting tangents at special points on a curve given by means of an analytic expression. This kind of analytic representation allowed an expansion of the universe of curves to which tangents could be plotted. Thus, a local organization with the following components emerged:

*A curve described in terms of an equation, and an operational field consisting essentially of "deriving" the equation and making this "derivative" equal to zero.*

Any text dealing with the history of calculus will reveal that this is the kind of mathematical tool used by Fermat. This local domain of intelligibility is bound to the context provided by analytic geometry; it generalizes the problem of plotting tangents by means of a new representation of the geometric object that is to be manipulated – the curve. The exploration of the derivative in this local domain does not in principle require a formal definition of the concept being explored; rather, it is on the basis of the local domain that the concept is constructed. In the case at hand, plotting tangents to convex curves, the operational field is sufficient. A problem arises, however, when we try to plot a tangent at a point of inflection[3]. There, the operational field indicates that the tangent is a straight line that intersects the curve, but the concept of a tangential straight line through the generalization of tangents to convex curves - "opposes" its acceptance as such. This tension between the operational field and the tangent concept is strengthened by the ultimate independence that the operational field acquires from the original conception of tangents. We are therefore forced to modify the concept to make it fit new situations. In this way, the original concept gradually acquires a higher level of organization, it becomes more abstract, it gains its independence from the context from which it arose, and a greater degree of compatibility with the operational field emerges. Ultimately, this activity gives rise to what we call the concept of a derivative.

**OUR THIRD EXAMPLE: EUCLIDEAN ORGANIZATION**

The theory of knowledge proposed by Aristotle defined mathematical objects as a result of the abstraction of objects from nature. The thesis that any body of knowledge, over the course of its development, tends towards a search for its principles translated, after considerable-effort, into the classical Euclidean system. Geometric intuition (endnote vi) is developed from a set of internalized actions, wherein the physical object is captured by means of language. Mathematized objects may have a degree of logical autonomy but ontologically they remain dependent on physical objects and,

[3] Readers well versed with 17th century calculus may find our exposition in this section "ahistorical" because we are attributing derivatives or differentiation to Fermat a bit strange. Again the point of this exposition is to explore the consequences of the problems arising from constructing tangents at points of inflexions. Some historians claim that Leibniz dealt with this matter quite correctly in his very first paper on the calculus in 1684.

consequently, are forced to respect the limits imposed on the latter. From this perspective, the object "mathematical magnitude" remains subordinated to the object "physical magnitude," in such a way that there is a permanent, ontological control over the objects of Euclidean mathematics. Postulates must be "self-evident," a property they inherit from the material conditions whence they come. These considerations help understand why Euclid made a major effort to keep the postulate of parallels on the sidelines of his geometrical development. In fact, saying that a single parallel passes through a point external to a straight line is tantamount to making a statement that evades verification through the corresponding physical object. Since we are studying a postulate, we have to keep the ontological problem in mind and, consequently, the limits imposed on mathematized objects. As is well known, the first thing Euclid did was to replace the above version of the parallel postulate with a version that did not explicitly mention the infinite. The cost was very high: the new version was very lengthy and complicated, and it gave every impression of being a proposition that could be deduced from the remaining postulates. It reads as follows:

*If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles* [Heath: Euclid. The Thirteen Books of the Elements, 202]

This gave rise to a desire for simplification on the part of scholars, who hence decided to try to prove this new proposition instead of accepting it as "self-evident". The richness of symbolic actions is greater than that produced by dealing with ontologically controlled objects, as in the case of Euclidean geometry. At that time, however, there was no awareness of that fact, as evidenced by the continued search for a proof, which even though it took place at the formal level, attempted to leave evidence of the Euclidean nature of space. The formal structure was not thematized (see endnote iv) as an object of investigation. The history of these efforts is very, very long, covering more than twenty centuries – an unequivocal guarantee that we are facing a major problem of knowledge. The outcome of this process has been the development of non-Euclidean geometries.

**STRUCTURES AS OBJECTS**

The approach to the parallel problem changed in the early 18th century. The new focus replaced the fifth postulate with one of the forms adopted by its negation - that is, more than one parallel passes through a point external to a straight line. In addition, the other postulates are maintained. The work of Saccheri (1667-1733), and, to a lesser extent, that of other geometers, took place during this period. The aim was to develop the consequences of the new axiomatic system until a contradiction emerged. The contradiction would be attributed to the presence of an absurd hypothesis (i.e., a negation of the fifth postulate). The metaphysical hypothesis - that is, the Euclidean nature of space - continued to weigh heavy in the interpretation: the contradiction would come from the absurdity, as opposed to the inconsistency, of a hypothesis that went against the geometers' intuitions about space. The mathematical objects involved up to this point were the product of an abstraction of the material objects that the former group of objects aspired to model; hence the metaphysical assertion about the geometrical structure of space. As the mathematicians' reflections fueled their interest in the system of relations provided by the axiomatic system, a weakness became apparent in the ontological control over the mathematical object and, at the same time, there was a strengthening in the role of the logical structure of the system as a whole. Thus, the object ceased to be seen as the result of an empirical abstraction and was gradually seen as determined by the system of relations in which it was involved. It was there, in the system of relations that the new mathematical meaning of the terms was created. This situation allows the realist conception of Euclidean mathematics to be contrasted with the constructivist position from which mathematics arises as a result of the constructive activity of the subject and not as a mechanical reflection of a reality that transcends it. Mathematical objects are thus conceptual objects that serve as substrates and organizing nuclei of mathematical activities.

If these geometers, Saccheri included, assumed that the consistency test implied the Euclidean nature of space, it was because they had unconsciously changed the rules of the game; indeed, they had chosen a total symbolic control of the objects. Underlying all this was the hypothesis that the formal structure of the Euclidean system could impose its dictates on ontology. This displacement in the conception of mathematical activity definitively marks the thematization of the structure as an object for study. Gauss indicated this change in the rules in explicit terms. He separated the problem of the consistency of the axiomatic system from the problem of the nature of physical space. In 1824, regarding the sum of the angles in a triangle, he wrote the following to his friend Taurinus:

*...but the situation is quite different in the second part, that the sum of the angles cannot be less than 180 degrees ; this is the critical point, the reef on which all the wrecks occur [...] the assumption that the sum of the three angles is less than 180 degrees leads to a curious geometry, quite different from ours [the Euclidean] but thoroughly consistent which I have developed to my entire satisfaction, so that I can solve every problem in it with the exception of the determination of a constant, which cannot be designated a priori […] the theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. For example, the three angles of a triangle becomes as small as one wishes, if only the sides are taken large enough; yet the area of the triangle can never exceed a definite limit regardless of how great the sides are taken [...] all my efforts to discover a contradiction, an inconsistency, in this non-Euc1idean geometry have been without success, and the one thing that is opposed to our conceptions is that, if it were true, there must exists in space a linear magnitude determined for itself (but unknown to us). But it seems to me that we know [...] too little about the true nature'of space, to*

*consider as absolute impossible that which appears to us unnatural...* [Greenberg, 1974, pp.142-144]

One of the apparently absurd results refers to the formula we use to calculate the area of a triangle. The assumption that more than one parallel passes through a point external to a straight line means that, given a triangle with angles that measure (in degrees) a, b, and c, its area is calculated by means of the formula: Area=k (180-(a+b+c)) where k is a positive constant that cannot be determined on an a priori base. It is c1ear from this formula that unlike what occurs under Euc1idean geometry, the area of a triangle depends on the length of its sides, just as Gauss explained in his letter to Taurinus. Gauss was right to appeal for calm. The result is astonishing. It is interesting to point out that Wallis tried to prove the fifth postulate, adding to the first four postulates the additional hypothesis that in geometry there could be triangles with arbitrarily large areas. The hyperbolic area formula discovered by Gauss very c1early demonstrates where Wallis's mistake is to be found. We can explain the illegitimacy of his hypothesis by progressing to a higher level of organization in the field of geometry, such as the one Gauss attained. Non-Euclidean geometries "erased" the ontological problem from mathematics. The concept of math model came into the front line. But we might have to take into account that the word "model" can be interpreted in two different ways: first, as an "ontological model" as it was interpreted along the struggle with the 5th postulate.

This is almost the same viewpoint, today, among natural scientists who believe that science is approaching gradually a better isomorphism between its models and reality. This is the "naive" viewpoint. Second, a model as a formal description of the scientists experience with reality. The semantic of these models come from the correspondence between the model and the experiments. These are clearly not the same. Of course, one can work "forgetting" these viewpoints but then, you can enter a maze like it was the case with non-Euclidian geometry, Saccheri for instance, who was absolutely certain of the Euclidian nature of space. Not that the geometry provided a model. Here we can see a deep obstacle to the inauguration of non-Euclidian geometry. The key was the distinction between ontology and consistency, first and then between ontology and the notion of a posteriori model. In making these considerations we can revisit the *Theorema Egregium* of Gauss (see endnote v). We conjecture based on historical evidence that Gauss saw beyond "the Boetians" as he used to say. Gauss in his letter to Taurinus wrote:

*"All my efforts to discover a contradiction, an inconsistency, in this non-Euclidean geometry have been without success, and the one thing that is opposed to our conceptions is that, if it were true, there must exists in space a linear magnitude determined for itself (but unknown to us)"*

Putting these lines besides the Egregium, it seems to us that Gauss knew then, that curvature was the key to understanding that problem. Measuring distances is the instrumental activity to "discover" the nature of a (math) space; it is the Egregium that allows us to write this statement. With the Gaussian curvature, we can re-define the question by the Euclidian nature of space. In fact, the presence of curvature solves simultaneously the ontological and the consistency problem as present in this quest for the nature of space. Curvature poses the problem at a higher level and suggests the transit from (something close at least) the concept-image and concept-definition of a notion of space. The next example shown below is a response to the separation between arithmetic and geometry caused by the Pythagorean crisis.

## THE CONCEPT OF QUANTITY FOR STEVIN

The mathematical work of Simon Stevin that contains most of his theoretical contributions is L 'Arithmetique, published in 1585 (see endnote vii). In the first volume Stevin presents an extension of the concept of number that is only possible after an explicit rupture with the Euc1idean conception. By means of this new concept of number, Stevin tries to give foundation to the procedures of arithmetical calculation with his recently introduced decimal notation (see endnote viii). Stevin begins his treatise with the following definitions:

*Definition 1: Arithmetic is the science of number*

*Definition 2: Number explains the quantity of each thing*

Definition 2 contains the great change that Stevin introduced in the conception of number

[Stevin, L' Arithmetique, p. 1] concerning Euclidean mathematics. For Aristotle, number is a quantity (a class of them) X, while according to Stevin, number is the means by which quantity is made evident (quantity being a property of things). Let us note this difference: For both authors quantity is a concept that is abstracted - at a first level of representation - from certain properties of material objects (their "quantitative being"), but while Aristotle places number and magnitude at this first level, Stevin passes on up to a second level of representation in which the number "speaks" of the preceding level. Number for Stevin does not only refer to the numerable quantity, but also the measurable quantity, this implies a change in the actions associated with number: it is not only used to count but also to measure.

In addition to this new concept of number, whose objective was to establish identification in the treatment of discrete and continuous quantities, Stevin has to explicitly indicate important differences with respect to the number in Greek mathematics

a) Unit (numerical or geometric, now treated as identical) is number.

b) Unit is limitlessly divisible.

c) Parts of unit are, in turn, numbers.

In a direct fashion Stevin obtains from these three premises an extension of the numerical domain that came to include unit and the fractions of unit. But this was not the only extension he introduced since he generalized the limited closure of Greek arithmetical operations to the operations of the extraction of roots, thus incorporating root numbers (rational and irrational) and all those numbers that result from algebraic operations with positive numbers.

In the second place, Stevin disposes of the continuous-discrete quantity dichotomy by rejecting "discreteness" of number as a characteristic of its essence: "Number is not discontinuous quantity" [L 'Arithmetique, p. 2]. Number, as an isolated entity, is "continuous" in the Aristotelian sense, since it can be divided indefinitely and, in any case, it inherits the characteristics of continuity or discreteness of the “thing that it is quantifying. For example, if we talk of one horse, the number one is discrete, however, if we talk of one yard, the number one is continuous (endnote ix). With this formulation, continuous and discrete cease to be ontological categories; the discussion of this point is circumscribed to the field of mathematics where they are the circumstantial properties of the objects quantified.

Third, unit no longer has the privileged character it had in Greek mathematics, as the principle of generation of number. With his formulation, Stevin has to renounce the possibility of having an absolute generating principle and was obliged to look for an extra logical foundation for his concept of number, that is, a foundation external to mathematics, located in the physical context: the quantity of each thing. Stevin's work is marked by the predominance of arithmetic operations and concrete actions to give reality to number. We can see various points in his work - not where he lays out principles but in the adjacent arguments - where this can be seen. Let us look at these points.

a) The essence of the number lies in its operations. - The first indication that Stevin's concept of number is conditioned by the operations that can be carried out with it, is to be found in a passage in L 'Arithmetique where Stevin argues in favor of the division of the unit. To deny the divisibility of the unit, Stevin says, is to limit the nature of the number, the essence of which is shown in the arithmetic operations that many authors had carried out by dividing the unit (as Diophantus does) [L 'Arithmetique, p. 2].

The above statement gives number an operational existence, that is, it is the operations that we can carry out with numbers that determine its nature. But where are arithmetic operations sustained? Let us imagine how Stevin might have argued this: Number explains the quantity of each thing. Arithmetic operations, as relations and transformations of numbers, explain the actions or transformations that are carried out with things (insofar as they can be quantified). Therefore, arithmetic operations are sustained by the actions that are carried out on quantities. The former, in turn, constitute the essence of the number in the same way that the actions of measurement, comparison, division, transformation, etc. are what give meaning to quantity.

b) Closure of the numerical domain regarding algebraic operations. - A very important

step towards the widening of the numerical domain is to consider that results of algebraic operations carried out with numbers are numbers. Stevin argues this point in two ways. The first, based on the practice of measurement, consists of exhibiting geometric magnitudes that can be associated with such results. The second argument - used repeatedly by Stevin - has an extra-logical character and was harshly criticized in its time. Let us look at this second kind of argument as it is found the first time it was presented, when Stevin argued that unit is a number.

**THAT UNIT IS NUMBER** (see endnote x,xi). *The part is the same matter of its whole. Unit is a part of the multitude of units. Therefore unit is the same matter of the multitude of units; But the matter of the multitude of units is number Therefore the matter of unity is number.* [L'Arithmetique, p. 1].

This argument presents a clear ambiva1ence regarding its level of abstraction. The first premise, "the part is of the same matter of its whole", refers to material objects, while the second, "Unit is part of the multitude of units", refers to mathematical, and thus abstract, objects. The unrestricted step from one level to another shows the normative underpinning that physical reality gives to the concept of number for Stevin. Stevin sustained his concept of number on the basis of the operations that can be carried out with it, and the operations themselves, on the actions that can be carried out on matter (to the extent it can be quantified). Stevin's unit is not only the result of an abstraction from objects as quantities, but principally of all abstraction from the coordinated actions that are carried out in the measuring these objects process.

## CONCLUSION

We have provided examples of key moments in the development of mathematics that support our thesis that the nature of mathematical objects is co-substantia1 with the operational inventions that accompany them and that determine the normativity to which they are subjected. The contexts of discovery and of justification cannot be separated, as logical empiricism attempted to do. On account of its suggestive power, we would like to recall what Jean Cavaillés (1947) wrote at the end of his book on logic and the theory of science (p. 43):

> *The theory of proof demands an ontology. However, the patterns and conceptual reference points which the mathematician uses as mediating elements in his work, do not constitute an ontology.*

The history of non Euclidean geometries is enough to show us that such an interpretation would be inappropriate: initially, geometric intuition comprises a set of internalized actions. Since actions represented mentally are richer than sensory-motor actions (see endnote xii), however, they give rise to co-ordinations that supersede intuitive space and provide the starting point of representative space. The construction of different geometries demonstrated the inability of intuitive space to exhaust the operational activity of the subject. Mathematics, as several scholars have suggested, is not merely a body of knowledge; it is, above all, an activity. The different levels of organization and their networks of meanings make up the substrate of this activity in existence at different times during the development of mathematics. These networks allow interpretations of the phenomena that become objects of study. In this way, the development of the discipline corresponds to the construction of theories with greater explanatory capacities.

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ENDNOTES

i. We thank Professor Gil Henriques of the University of Geneva for an enlightening discussion on various sections of this paper. The paper is dedicated to the memory of Guillermina Waldegg.

ii For example, Edwards, 1979, pp. 122-125

iii For example, mathematical magnitude can only be potentially infinite (see Moreno and Waldegg, 1991).

iv "The change from usage, or implicit application to consequent use, and conceptualization constitutes what has come to be known under the term 'thematization'" [Piaget and García (1989), p. 105]

v See Bonola, 1955 pp. 43-51 as well as Boí, 1995, pp. 20-27

vi Not in the sense used by Brouwer, but in that of Piaget 1950

vii The first part of this volume (L 'Aríthmetique) contains definitions, operations and a translation with comments of the six first books of Diophantus's Algebra. The second part (La Practique d'Aríthmetique) contains solved problems, La Dísme and a treatise on incommensurable magnitudes

viii During the same year (1585), Stevin published a short pamphlet called La Dísme on decimal notation and its arithmetic, including the treatment of decimal fractions. The whole pamphlet was included in the first edition of L 'Aríthmetique

ix Nombre est cela, par lequel s'explique la quantité de chascune chose

x Aristotle: Categoríes 1 b,25

xi QUE L'UNITE EST NOMBRE (Capitals in the original)

xii See Piaget (1970), p. 41