From Problem Solving to Modelling- A meta-analysis

Nicholas Mousoulides[1], Constantinos Christou[2], Bharath Sriraman[3]

Abstract: Mathematical modelling is a complex mathematical activity, and the teaching and learning of modelling and applications involves many aspects of mathematical thinking and learning (Burkhardt & Pollak, 2006; Niss, 1987; Kaiser, Blomhøj & Sriraman, 2006). An increasing number of mathematics education researchers have begun focusing their research efforts on mathematical modelling, especially at the school level. It is not simply a case of researchers changing their agenda, as much as a growing awareness among the mathematics education community of the need for change (Lesh, Kaput & Hamilton, 2007; Sriraman & Lesh, 2006). More than 25 years ago, a NSF funded project investigated the question: what is needed, beyond having a mathematical idea that enables students to use it in everyday problem solving situations? (Lesh, Landau & Hamilton, 1983). The answer to this question has begun to emerge after 25 years of systemic work in the domain of modelling. In this paper, we chronicle and meta-analyze the emergence of modelling perspectives around the world from the genre of problem solving research and synthesize major strands in the extant literature.

Keywords: design research; learning; modelling; modelling design; modelling research review; problem solving; teaching

[1] Faculty of Education, University of Cyprus
[2] Faculty of Education, University of Cyprus
[3] Dept. of Mathematical Sciences, The University of Montana

Introduction

An increasing number of mathematics education researchers have begun focusing their research efforts on mathematical modelling, especially at the school level. This is evident in numerous research publications from groups of researchers in Australia (English, Galbraith and colleagues), Belgium (Verschaffel and colleagues), Denmark (Niss, Blomhøj and colleagues), Germany (Blum, Kaiser and colleagues), Netherlands (de Lange and colleagues) and the U.S (Lesh, Schoenfeld and colleagues), among many others. The question then is how well prepared are learners today to solve the problems that they will encounter beyond school, in order to fulfil their goals in work, as citizens and in further learning? (Christou, et al., 2005; OECD, 2004; Doerr & English, 2003; Schoenfeld, 1992). How can students work with problems that are less obviously linked to school mathematics and require students to deal with unfamiliar situations by thinking flexibly and creatively (Lesh& Doerr, 2003a, 2003b)? It is necessary to distinguish *problem-solving* activity as described by Schoenfeld (1992) and other researchers (English, 2003) from the activity of solving traditional word problems. The latter, presents problems that are simplified forms of decontextualized world based situations that serve a specific and minor purpose, being the exercise for a specific type of mathematical learning, such as addition or subtraction (Wyndhamm & Saljö, 1997). The practice of word problem solving in school mathematics hardly matches this idea of mathematical modelling and *mathematization*, which is the structuring of reality by mathematical means (Freudenthal, 1991). Student work with that type of solving problems is absent of heuristics and mathematical strategies and the result is a prevalence of mechanical and mindless solutions (Greer, 1997). A second, probably worse, consequence of student work in solving traditional word problems is the absence of high level cognitive and metacognitive processes involved. This absence is forwarding students to look for key words and employ direct translation strategies to solve a problem (Schoenfeld, 1992).

A modelling perspective to problem solving leads to the design of an instructional sequence of activities that begins by engaging students with non-routine problem situations that elicit the development of significant mathematical constructs and then extending, exploring and refining those constructs in other problem situations leading to a generalizable system (or model) that can be used in a range of contexts (Lesh & Doerr, 2003a, 2003b; English & Doerr, 2004). For instance, in problem solving activities, referred to as model eliciting activities (Lesh & English, 2005; Lesh & Sriraman, 2005), the products that students produce go beyond short answers; they include sharable, manipulatable, modifiable, and reusable conceptual tools (e.g. models) for constructing, explaining, predicting and controlling mathematically significant systems (Lesh& Doerr, 2003a,2003b).

Students' descriptions, explanations, and justifications form an integral component of the models the students produce. In contrast to many of the problem situations students meet in school, modelling activities are inherently social experiences, where students work in small teams to develop a product that is explicitly sharable. Numerous questions, issues, conflicts, resolutions, and revisions arise as students develop, assess, and prepare to communicate their products (English & Doerr, 2004; Lesh & Sriraman, 2005). In an attempt to review the related literature and provide a coherent state of affairs, we organize the literature review into three major discussion strands. The first major strand situates mathematical modelling as being a problem solving activity and we discuss issues related to the teaching, learning and assessment of modelling. The second major strand presents some basic principles for designing and implementing modelling activities and finally the third strand discusses the benefits for students and teachers in working with thought revealing modelling activities.

Mathematical Modelling as a Problem Solving Activity

Inadequacy of Traditional Approaches

A number of researchers raise the question of the appropriateness of current teaching approaches in teaching mathematics and in mathematical problem solving in particular (Doerr & English, 2003; Blum & Niss, 1991; Sriraman & Lesh, 2006). The inadequacy of traditional approaches is even worse in the case of students' work with problems that are less obviously linked to school mathematics and require students to deal with unfamiliar situations by thinking flexibly and creatively (Lesh & Doerr, 2003a, 2003b). A necessary distinction should be made between 'problem-solving' activity, as proposed by Polya (1973) and Schoenfeld (1991) and relates to Dewey's *reflective thinking* and the activity of 'solving problems', the traditional use of word problems in a school environment. Polya (1962) stressed that "in solving a word problem by setting up equations, the student *translates* a real situation into mathematical terms: he has an opportunity to experience that mathematical concepts may be related to realities, but such relations must be carefully worked out" (p. 59). National Research Council documented that mathematical problem solving rather deals with data, and observations from science… is far more than just calculation or deduction; it involves observations of patterns, testing of conjectures, and estimation of results (NRC, 2001). Word problems appear in most textbooks usually represent recontextualized forms of decontextualized descriptions of everyday life situations that serve a specific purpose; being exercises for specific types of mathematical learning, such as addition or subtraction (Wyndhamm & Saljö, 1997). Hiebert et al., (1996) pointed that solving such word problems cannot prepare students for everyday life, since students are not able to transfer the specific domain-related knowledge (mathematics) and also more general problem- and solution-related skills. Their recommendation was to base instructional design on problem solving in out-of-school situations, rather than on problem-solving models of limited applicability (Hiebert et al., 1996).

A significant number of recent research studies pointed out that the practice of word problem solving in school mathematics hardly matches this idea of mathematical modelling and mathematization (Reusser & Stebler, 1997, Kaiser, Blomhøj & Sriraman, 2006; Kaiser & Sriraman, 2006; Mousoulides et al., 2006). As a result, students readily 'solve' unsolvable, even absurd, problems if presented in ordinary classroom contexts. Students do not have opportunities to do mathematics; students look for key words, employ direct translation strategies when solving stereotyped word problems and their ability in solving these problems is influenced by contextual information (Greer, 1997; Vershaffel, De Corte & Lasure, 1994; Yoshida, Verschaffel & De Corte, 1997).

The role of context in mathematical modelling

The role of context is very important in mathematical modelling, since modelling requires a context in which to 'frame' the problem and 'develop' the mathematics. This meaning of contextual teaching and learning is much closer to the notion of *situated learning* (de Lange, 1992). Researchers have documented a number of different functions a "realistic context"can play in problem solving. De Lange (1987) listed the following functions: enabling concept formation, facilitating model formation, providing a wider range of utility and more interesting practice problems. Similarly, DaPueto and Parenti (1999) reported the following factors for using a realistic context in mathematical problem solving: (a) Facilitating collaboration, interaction and contribution of students who have different *styles* of exploration/understanding/use of concepts, different *levels* of formalized knowledge, etc, (b) helping more balanced development of *reflective learning and experiential learning,* and (c) facilitating the design and restructuring of the *schemata* through which knowledge is organized (p. 7).

A number of researchers commented on the importance of contextual based problem solving.

Bohl (1998) identified the lack of relevance as a critical factor in engaging students, and certainly, the use of contextual settings has the potential to address that. Pace (2000) further pointed that students need to experience real-world situations in the chosen context before they can create models for solving world based problems. Contextual based teaching encourages the development of an active learning environment which in turn generates better mathematical understandings by the students (Pace, 2000). In line with previous findings, Gravemeijer and Doorman (1999) stressed that research on the design of Realistic Mathematics Education (RME) based activities has shown that the use of personalized contexts improved word problem solving by increasing the meaningfulness of contexts and enhancing student motivation. Gravemeijer and Doorman (1999) also claimed that "well-chosen context problems offer opportunities for the students to develop informal, highly context-specific solution strategies" (p. 117).

Research has also indicated a number of possible limitations related to the use of contexts in problem solving. The choice of context may be a turn-off to particular students or be so familiar as to inhibit some approaches judged as unreasonable (Choi & Hannafin, 1997). McNair (2000) pointed that if the situation and experience used are not familiar to the students, then instruction may degenerate into the presentation of abstract ideas related to the context. Finally, Resnick (1988) addressed that there is a need for finding ways to create in the classroom situations of sufficient complexity and engagement that they become mathematically engaging contexts in their own right.

Affective Factors and Mathematical Modelling

The importance of the affective domain and students' communication in classroom work is stressed in a number of research studies (Lesh & Doerr, 2003a, 2003b; Gravemeijer, 1997; Verschaffel et al., 1997). Yoshida, Verschaffel & DeCorte (1997) reported that mathematical problem solving is now also focussed to student attitudes and beliefs and their capacity to apply their mathematical knowledge in authentic, non-routine problems. The importance of considering student conceptions is also pointed by Confrey & Doerr (1994), who argued that the use of learner-centered modelling tools and approaches can create positive beliefs in the mathematics classroom. The use of realistic mathematical modelling problems can enhance student sense-making, while bringing real-world situations into school mathematics is a necessary condition to foster a positive attitude towards mathematics.

While the use of modelling as a problem solving activity can positively change student beliefs towards mathematics, a significant hazard in teaching mathematical modelling is teachers' pre-conceived beliefs, and the projection of those beliefs onto their students. Verschaffel and his colleagues (1997) reported a strong and resistant tendency among teachers to exclude real-world knowledge when teaching arithmetic word problems. Teachers' pedagogical beliefs stressed that non routine problems are unimportant and the goal of teaching word problem solving is to find the correct numerical answer. Verschaffel and his colleagues (1997) concluded that these teacher attitudes and beliefs about the significance of real world knowledge in problem solving have a negative impact of their teaching practices and consequently on their students' learning processes and outcomes. The demand on teachers to adopt the modelling approach in their problem solving teaching and their resulting beliefs is articulated by Blum and Niss (1991). They stressed that problem solving is more demanding, requiring additional qualifications. Moreover, many teachers do not feel able to deal with non routine problems and very often teachers simply do not know how to prepare their teaching as such.

Assessment in mathematical modelling

Niss (1987) pointed that assessment of modelling could be problematic, since modelling is difficult to assess, let alone test, by traditional evaluation tools. Niss (1993) further clarified that assessment takes time and cannot be standardised. It does not imply that assessment cannot be exercised on a sound foundation of reflection and reasoning and articulate criteria and be subject to clear communication (Niss, 1993).

A number of different types of assessment being used to evaluate students' modelling abilities and understanding of models are found in a review of the literature. Crouch and Haines (2004) used a multiple choice format, in developing several test questions related to modelling, Kitchen (1993) proposed questions requiring students to set up a model, interpret a solution, or criticize a model based on the data, while Bell and colleagues (1992) and Hjalmarson (2005) suggested the use of an analytic scoring scale, by assigning point values to various dimensions of the modelling work.

Modelling Activities : Design and Research

Modelling Processes in Problem Solving

The modelling approach to problem solving suggests that there is not a single powerful procedure between givens and goals and a set of "strategies" for overcoming any difficulties in this procedure. Indeed, the modelling approach indicates a number of trial procedures between givens and goals in order to succeed a solution. Problem solving includes a number of iterative cycles, in which students move from givens to goals, go back and again moving towards goals to test their hypotheses, refine their results and to improve their solutions (Lesh & Doerr, 2003).

A number of relevant works (Lesh et al., 2003; Blum & Niss, 1991) have documented the different processes involved in mathematical modelling as problem solving activity. In particular, students engage in the following processes: (a) Understand and simplify the problem. This included understanding text, diagrams, formulas or tabular information and drawing inferences from them; demonstrating understanding of relevant concepts and using information from students' background knowledge to understand the information given. (b) Manipulate the problem and develop a mathematical model. These processes included identifying the variables and their relationships in the problem; making decisions about variable relevancy; constructing hypotheses; and retrieving, organising, considering and critically evaluating contextual information; use strategies and heuristics to mathematically elaborate on the developed model. (c) Interpreting the problem solution. This included making decisions, analysing a system or designing a system to meet certain goals, and diagnosing a malfunction and proposing a solution, and (d) Verify, validate and reflect the problem solution: This included constructing and applying different modes of representations to the solution of the problem; generalize and communicate solutions; evaluating solutions from different perspectives in an attempt to restructure the solutions and making them more socially or technically acceptable; critically check and reflect on solutions and generally question the model (Blum & Kaiser, 1997; Lesh& Doerr, 2003a, 2003b).

Student and Teacher Models

A model is an internal conceptual system plus the external representations of that system used to interpret other complex systems (Lesh & Doerr, 2003a, 2003b; Lesh, Doerr, Carmona & Hjalmarson, 2003). Typically, this definition of model has only been used in reference to student or teacher thinking and learning (e.g., Doerr & Lesh, 2003). To provide a parallel construct at the researcher level, a design experiment carried out from a models and modelling perspective (a modelling design experiment) should be consistent with this definition. The design tested by the experiment encompasses two parts (similar to a model). Namely, the design includes theoretical assumptions (i.e., researcher-level conceptual systems about mathematical knowledge, models, teacher development, etc.) and external artifacts (i.e., representations of the researcher-level conceptual system in the form of interventions, curriculum, etc.) (Lesh & Doerr, 2003a, 2003b; Lesh & Sriraman, 2005). Models consist of an internal conceptual system and external artifacts or representations (Lesh& Doerr, 2003a,2003b; Lesh et al., 2000). In addition, models incorporate a number of external representations (e.g., a graph, a table). In constructing models, students identify, select and collect relevant data, express limitations and conditions of a model, interpret the solution in context, communicate effectively and describe situations using a variety of representation forms.

Teacher-level models for teaching mathematics include not only the mathematical components of student models, but also pedagogical and methodological elements for helping students develop their own mathematical models (Doerr & Lesh, 2003). As with student models, teacher models consist of two parts: internal conceptual systems and external artifacts. However, from a modelling perspective, there is no separation between the external artifact and the conceptual systems. Rather the two are interconnected in one model. As with student mathematical models, external artifacts change as internal conceptual systems change and vice versa (Lesh & Doerr, 2003a, 2003b).

Principles and assumptions about student-level and teacher-level mathematics learning and development should also apply to researcher-level modelling of the other two tiers. One assumption is that researcher designs develop along multiple dimensions just as student models develop along multiple dimensions (Lesh, 2002). For example, student models may move from unstable to stable or from simple to complex. As researchers study a design, unstable early assumptions are repeatedly tested and become more well-developed and stable. Some assumptions or artifacts may be revised throughout the study and eventually may stabilize at some point for the particular situation. When the artifacts are transported to another situation, they may become unstable again. Student models may be very simple at the start of their solution process.

Characteristics of Modelling Activities

The different tools being designed and created to facilitate students' and teachers' externalization of their thinking and understandings of problem situations aim to elicit their thinking and thus researchers are referring to these tools as model eliciting activities (Lesh et al., 2003; Lesh & English, 2005; Lesh & Sriraman, 2005). Among the central characteristics of these activities are: a. to develop a model that describes a real-life situation, b. the provided models to encourage the solver to describe, revise, and refine their ideas; and c. the models encourage the use of representational media to explain (and document) their conceptual systems. Model-eliciting activities can be designed to lead to significant forms of learning because they involve mathematizing –by quantifying, dimensioning, coordinating, categorizing, algebraizing, and systematizing relevant objects, relationships, actions, patterns, and regularities. An example of a model eliciting activity for students is intended to reveal the way students are thinking about a real life situation that can be modelled through mathematics. The solution calls for a mathematical model to be used by an identified client who needs to implement the model adequately. As a result, students must clearly describe their thinking processes and justify not a single solution, but rather all (or most of) the optimal and appropriate solutions (English, 2003). Students' engagement with such mathematical tasks results in developing math concepts through the need to develop powerful math ideas in order to solve a problem. Thus, they are given a purpose (and End in View) (English & Lesh, 2003) to develop a mathematical model that best explains, predicts, or manipulates the type of real-life situation that is presented to them. In this way, model-eliciting activities allow students to document their own thinking and learning development. The aim of a modelling activity includes problem specification and validation, engage in critical usage of modelling, participation and communication skills; foster creative and problem solving attitudes, activities, competencies; provide the opportunity for students to practice applying mathematics that they would need as individuals in society; to contribute to a balanced picture of mathematics; to assist in acquiring and understanding mathematical concepts (Battye & Challis, 1997).

Types of the Modelling Activities Product

Model Eliciting Activities include three types of products; tools, constructions and problems.

(1) Product as a tool. Tools fulfil a functional or operational role and they include: (a) *Models*. Models are used for ranking items, people and places; determining loan payments and may form the based of complex systems such as company's financial operations, (b) *Descriptions and Explanations.* Descriptions and Explanations illustrate and verify the results of an experiment or investigation or may describe why something that appears superficially correct is mathematically incorrect, (c) *Designs and Plans.* Used in all walks of life, designs and plans must meet detailed and complex criteria and must incorporate appropriate mathematical and representational systems, and (d) *Assessment Instruments.* They are used in a wide range of contexts such as assessing learner's progress, and selecting staff. They normally undergo rigorous development that incorporates cycles of testing, refining and applying (Lesh& Doerr, 2003a,2003b).

(2) Product as a construction. A construction normally requires students to use given criteria to develop a mathematical item. They do not define the nature of the product rather they set parameters for the design of the product. A construction can be in the form of: (a) Spatial Constructions, (b) Complex Artefacts. Inventions are a good example of complex artefacts. The criteria for their design frequently focus on deficits in existing artefacts or on perceived societal needs, (c) Cases. Cases make use of persuasive discourse to adopt a stance on an issue, to recommend one course of action over another, or to highlight an issue in need of attention. Cases are especially effective when they draw upon mathematical data to support

their claims and (d) Assessments. They are the products of applying an assessment tool. Such products can serve a number of purposes and usually suggest or imply courses of action (Lesh& Doerr, 2003a,2003b).

(3) Problem as a product. The ability to pose problems is becoming increasingly important in academic and vocational contexts. During modelling cycles involved in model eliciting activities students are engaged in problem posing, that is, they are repeatedly revising or refining their conception of the given problem. During the model eliciting activities, students find ways to judge strengths and weaknesses of alternative ways of thinking and whether a given response is appropriate and good enough (Lesh& Doerr, 2003a,2003b; English & Lesh, 2003).

Principles for Modelling Activities

One defining characteristic of a design experiment is that the researchers create, test, and modify a design within the context of use (Design-Based Research Collective, 2003). For example, the researchers may be testing a new curriculum or teaching method in a classroom (e.g., Erickson & Lehrer, 1998; Verschaffel et al., 1997). This characteristic is consistent with model-eliciting activities that ask students to develop mathematical models to explain real-life situations. The development of a design or model is also often cyclic (Lesh & Lehrer, 2003). In a typical series of cycles, the student expresses thinking in some artifact or product, tests the artifact, and then revises the artifact. For example, a student creating a consumer guide for buying a car developed a spreadsheet for scoring characteristics of cars, asked other members of their group or class to test the appropriateness of their scoring guides (to test the product), and then revised the product based on testing results to improve their solution (improve the product) (Hjalmarson, 2005). The students' revisions are guided by a purpose (end-in-view) that describes the functions the final product should be able to perform (English & Lesh, 2003). Similarly, for modelling design experiments, researchers should have some end-in-view for the product under development. The end-in-view should guide researcher decision-making about revisions that are made to the product from research cycle to research cycle.

An important caveat is that for design experiments using a models and modelling perspective, the assumptions and understandings of the teachers (and researchers) may change throughout the study. It is imperative to document those changes as they are made (Lesh & Sriraman, 2005). Often, researchers are interested in the students' development of responses or in how student models change within a session or between modelling sessions. So, rather than studying fixed constructs or examining snapshots of constructs in isolation, researchers may be studying changes in constructs over time and across problems and individuals. Capturing change and the effects of change can be a goal of design experiments with a models and modelling perspective. So, both components of the design (theoretical assumptions and artifacts) will change just as for students' models both the internal conceptual system and the external representations change. This characteristic is another example of how the researcher-level design experiment should be consistent with the student-level.

For model-eliciting activities, a crucial component is the local context that situates the task. The context guides the students' development of solutions, aids in their decision-making about whether a way of thinking is "bad" or "good", and helps them place the end-in-view in a context that is real to the students (English & Lesh, 2003). The context situates the usefulness of the design and aids development since the final product should be useful in that context (Design-Based Research Collective, 2003). However, this does not suggest that the products are not generalizable to other situations (or contexts). As with model-eliciting activities where students develop a product for a particular client that is generalizable to other (similarly structured) situations, designs should also be generalizable to other educational situations. This proviso means that the researcher needs to outline precisely the conditions under which the design was used and possible modifications that may need to be made for the design to be appropriate for different situations (Design-Based Research Collective, 2003).

Collaboration is also a component of design experiments following the modelling perspective that parallels assumptions about student learning. Collaborators may include researchers, teachers and students proceeding along multiple levels of development similar to multi-tiered teaching experiments (Kelly & Lesh, 2000; Lesh & Kelly, 2000; Schorr & Lesh, 2003). Researchers need teachers to help design, test and implement products. Products should be developed with teachers' questions about their own practice in mind (e.g. personal meaningfulness), and researchers can provide resources to aid teacher development (Design-Based Research Collective, 2003). There may also be multiple teachers or researchers involved in the development of any product. This characteristic can aid the triangulation of interpretations about results and the generalizability of results if products have been tested in multiple contexts. Collaboration also aids the documentation of results by requiring that strategies or tools need to be communicated to other people for comment (e.g., individual teachers develop a ways of thinking sheet or concept map to share with the group) (e.g., Koellner-Clark & Lesh, 2003).

Appropriateness, Usefulness and Benefits of Modelling Activities

It is imperative that mathematics educators take students beyond the traditional classroom experiences, where problem solving rarely extends their thinking or mathematical abilities. There is a strong need to implement worthwhile modelling experiences in the elementary and middle school years if teachers are to make mathematical modelling a successful way of problem solving for students (Blum & Niss, 1991).

Modelling activities have been found appropriate to enhance students' and teachers' capacities to engage in problem solving, thereby laying the foundation for exploring complex systems (Lesh et al., 2003). These activities are highly innovative learning experiences (English, 2003). A number of related features have emerged, indicating a number of benefits of modelling activities, both for students and teachers. Modelling activities provide a pathway in understanding how students approach a mathematical task and how their ideas develop; these activities appear to provide a strong basis for teachers to interact with students in ways that would promote their learning (Doerr, 2006). In the following part of the literature review we summarize the benefits for students and teachers while working with thought revealing modelling activities, including benefits in student mathematical literacy and conceptual understanding, in student social development, in student metacognition, and in teacher pedagogical approaches and teaching practices.

Mathematical Literacy and Student Conceptual Understanding

Related research in mathematical modelling indicated that student work with modelling activities assisted students to build on their existing understandings and to be successfully engaged in thought-provoking, multifaceted complex problems (Lesh& Doerr, 2003a,2003b; English, 2003). Modelling activities set within authentic contexts, allow for student multiple interpretations and approaches, promoting intrinsic motivation and self regulation. A number of related research studies showed that the use of modelling activities encouraged students to develop important mathematical ideas and processes that students normally would not meet in the traditional school curriculum (English & Watters, 2004; Zawojewski, Lesh & English, 2003). The mathematical ideas are embedded within meaningful real-world contexts and are elicited by the students as they work the problem. Furthermore, students can access these mathematical ideas at varying levels of sophistication. Student work in modelling activities facilitates student development of generalizable conceptual systems. Students move beyond just thinking *about* their models to thinking *with* them for solving an important world based problem. English (2003) reported that there was considerable evidence that students' mathematical ideas had improved after they worked in a sequence of modelling activities. Mathematical language improved but also considerable fluency with the use of tables and data were acknowledged (English, 2003). However, there was an acceptance that students needed to know basic operations to be effective in these activities. Gravemeijer's and his colleagues (2000) related their work in connection with Freudenthal's (1971) comprehension of mathematics as an activity that involves solving problems, looking for problems, and organizing subject matter resulting from prior mathematizations or from reality. In solving modelling problems, students developmentally move from modelling situations in an informal way (model *of* the situation) to mathematize their informal modelling activity (model *for* reasoning). This transition from a *model of* to a *model for* is consistent with Sfard's (1991) process of reification.

Lesh and Doerr (2000) have pointed out that modelling activities can promote students' conceptual understanding. They clarified that in modelling activities students are not simply working with ready made models. Since models are interacting systems based in more complex conceptual systems, Lesh and Doerr (2003) claimed that models must be constructed in a meaningful way. This construction leads to conceptual understanding and mathematization (Lesh& Doerr, 2003a,2003b; Lesh & Sriraman, 2006).

Lesh and Harel (2003) and Harel and Lesh (2003) further stressed the importance of modelling activities by highlighting the importance of student conceptual systems. They documented that conceptual systems are developed first as situated models that apply to particular problem solving situations. Then, these models are gradually extended to larger classes of problems as they become more sharable, more transportable, and more reusable. The aforementioned features of modelling activities helped students be successful beyond problem situations for which models were created (Lesh & Harel, 2003).

Lesh and his colleagues (2003, 2006) and English (2003) investigated the role of modelling activities with regard to student algebraic reasoning. Lesh et al., (2003) reported that modelling activities provide opportunities for students to explore quantitative relationships, analyze change, and identify, describe, and compare varying rates of change, as recommended in the Grades 3-5 algebra strand of the Principles and Standards for School Mathematics (NCTM, 2000). In addition, English (2003) pointed that elementary probability ideas emerging when young students linked the conditions and constrains of problems (e.g., drug pain relief activity). The above research studies have also highlighted the contributions of these modelling activities to young students' development of mathematical description, explanation, justification, and argumentation. Modelling activities are inherently social activities, and as so, students engage in numerous questions, conjectures, arguments, conflicts, and resolutions as they work towards their final products. Furthermore, when they present their reports to the class they need to respond to questions and critical feedback from their peers (English & Watters, 2004; Zawojewski, Lesh & English, 2003; Lesh& Doerr, 2003a,2003b; Lesh & Sriraman, 2006).

An important parameter in students' work in modelling activities is students' use of their informal knowledge. Researchers have observed the interplay between students' use of informal, personal knowledge and their knowledge of the key information in the problem (Zawojewski, Lesh & English, 2003; Mousoulides et al., 2006). In a number of modelling activities, students' informal knowledge helped them relate to and identify the important problem information (e.g., understanding and interpreting the conditions for the solution of a problem). A number of researchers (Doerr, 2006; Doerr & English, 2003) also documented that students embellished their written reports with their informal knowledge and most importantly, many students recognized when their informal knowledge was not leading them anywhere and thus students reverted their attention to the specific task information (Doerr, 2006; Zawojewski, Lesh & English, 2003; Lesh& Doerr, 2003a,2003b).

Concluding Points

As the literature review has pointed out, it is vital that the mathematics education research community continually revisit the fundamental question : What does it mean for a younger student to understand models and modelling? We also need to take into account the claim (which we hear from many) that the nature of problem solving (and "mathematical thinking") has changed dramatically in the past 20 years (see Lester & Kehle, 2003; Lesh, Hamilton & Kaput, 2007, in press). We also think there is a real need for research about: (i) the nature of new "real life" situations where some type of mathematical thinking in needed for success, (ii) what it means to understand relevant knowledge and abilities, (iii) how these ideas and abilities develop, and (iv)how development can be documented and assessed. There is also a need to focus on a call for research which takes into consideration what we already know about concept development in children. Sriraman & Lesh (2006) argue that today, when some kind of mathematical thinking is needed to solve real problems, the products that need to be produced often involve much more than short answers to pre-mathematized questions. For example, they often involve developing conceptual tools (or other types of complex artifacts) which are designed for some specific decision maker and for some specific decision-making purpose – but which seldom are worthwhile to develop unless they go beyond being powerful for a specific purpose to being sharable with others and re-useable beyond the immediate situations in which they were first needed. Consequently, solution processes often involve sequences of iterative development→testing→ revising cycles in which a variety of different ways of thinking about givens, goals, and possible solution steps are iteratively expressed, tested, and revised (e.g., integrated, differentiated, or reorganized) or rejected. That is, the development cycles often involve a great deal more than simply progressing from pre-mathematized givens to goals when the path is not obvious. Instead, the heart of the problem often consists of conceptualizing givens and goals in productive ways.

References

Battye, A., & Challis, M. (1997). Deriving learning outcomes for mathematical modelling units within an undergraduate programme. In S. Houston, W. Blum, I. Huntley, and N. Neill (Eds.), *Teaching and learning mathematical modelling – Innovation, investigation and applications*. (Ch. 11) Chichester: Ellis Horwood.

Bell, A., Burkhardt, H., & Swan, M. (1992). Balanced assessment of mathematical performance. In R. Lesh and S. Lamon (Eds.), *Assessment of Authentic Performance in School Mathematics* (pp. 119-144), Washington, DC: American Association for the Advancement of Science.

Blum, W. (1993). Mathematical modelling in mathematics education and instruction. In T. Breiteig, I. Huntley, and G. Kaiser-Messmer (Eds.), *Teaching and learning mathematics in context* (pgs. 3-14). New York: Ellis Horwood.

Blum, W., & Kaiser, G. (1997). Vergleichende empirische Untersuchungen zu mathematischen Anwendungsfähigkeiten von englischen und deutschen Lernenden. *Unpublished application to Deutsche Forschungsgesellschaft.*

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects – state, trends, and issues in mathematics instruction. *Educational Studies in Mathematics*, *22*(1), 37-68.

Bohl, J. (1998). Problems that matter: Teaching mathematics as critical engagement. *Humanistic Mathematics Network Journal*, *17*, 23-31.

Bonotto, C., & Basso, M. (2001). Is it possible to change the classroom activities in which we delegate the process of connecting mathematics with reality? *International Journal of Mathematical Education in Science and Technology*, *32*(3), 385-399.

Brousseau G. (1997). *Theory of didactical situations in mathematics*. The Netherlands: Kluwer Academic Publishers.

Burkhardt, H., & Pollak, H (2006). Modelling in mathematics classrooms. *Zentralblatt für Didaktik der Mathematik*, 38(2), 178-195.

Burton, L. (1997). Mathematics—communities of practice? Paper presented at *Meeting of the Research into Social Perspectives on Mathematics Education Group.* Birmingham: University of London, Institute of Education.

Choi, J., & Hannafin, M. (1997). The effects of instructional context and reasoning complexity on mathematics problem solving. *Educational Technology Research and Development*, *45*(3), 43-55.

Christou, C., Mousoulides, N., Pittalis, M., Pitta-Pantazi, D., & Sriraman, B. (2005). An empirical taxonomy of problem posing processes. *Zentralblatt für Didaktik der Mathematik (International Reviews on Mathematical Education), 37*(3), 149-158.

Clarke, M. D. (1996). The Changing Role of the Mathematics Teacher, *Journal for Research in Mathematics Education*, *28*(3), 278-308.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in education research. *Educational Researcher, 32*(1), 9-13.

Collins, A. (1992). Towards a design science of education. In E. Scanlon and T. O'Shea (Eds.), *New directions in educational technology* (pp. 15-22). New York: Springer-Verlag.

Confrey, J., & Doerr, H. (1994). Student modelers. *Interactive Learning Environments*, *4*(3), 199-217.

Crouch, R., & Haines, C. (2004). Mathematical modelling: transitions between the real world and the mathematical model. *International Journal for Mathematics Education in Science and Technology,35*, 197-206.

da Ponte, J. (1993). Necessary research in mathematical modelling and applications. In T. Breiteig, I. Huntley, and G. Kaiser-Messmer (Eds.), *Teaching and learning mathematics in context* (pp. 219-227). New York, NY: Ellis Horwood.

DaPueto, C., & Parenti, L. (1999). Contributions and obstacles of contexts in the development of mathematical knowledge. *Educational Studies in Mathematics*, *39*, 1-21.

de Lange, J. (1987). *Mathematics, insight and meaning – Teaching, learning and testing of mathematics for the life and social sciences*. The Netherlands: Utrecht University.

de Lange, J. (1992). Assessing mathematical skills, understanding, and thinking. In R. Lesh and S. Lamon (Eds.), *Assessment of authentic performance in school mathematics*. (Ch. 8) Washington, DC: AAAS Press.

Dekker, R., & Elshout-Mohr, M. (1998). A Process Model for Interaction and Mathematical Level Raising, *Educational Studies in Mathematics, 35*(3), 303-314.

Design-Based Research Collective (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher, 32*(1), 5-8.

Doerr, H. M. (2006). Examining the tasks of teaching when using students' mathematical thinking, *Educational Studies in Mathematics,* Online First. Retrieved 12.06.2006.

Doerr, H. M., & Lesh, R. (2003). A modelling perspective on teacher development. In R. Lesh and H. M. Doerr (Eds.), *Beyond constructivism: A models & modelling perspective on mathematics problem solving, learning & teaching* (pp. 125–140). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Doerr, H. M., & Tripp, J. S. (1999). Understanding how students develop mathematical models, *Mathematical Thinking and Learning, 1*(3), 231-254.

Doerr, H., & English, L. (2003). A Modelling perspective on students' mathematical reasoning about data. *Journal of Research in Mathematics Education, 34*(2), 110-136.

Eid, W. (1997). Verbalizing ideas in mathematics teaching. *International Journal of Mathematical Education in Science and Technology*, *28*(2), 161-183.

English, L. (2004). Mathematical Modelling in the Primary School. In I Faragher, R. McLean and M. Putt (Eds.), *Mathematics education for the third millennium: Towards 2010* (pp. 207-214). James Cook University: Mathematics Education Research Group of Australasia

English, L., & Lesh, R. (2003). Ends-in-view problems. In R. Lesh and H. M. Doerr (Eds.), *Beyond constructivism: A models & modelling perspective on mathematics problem solving, learning & teaching* (pp. 297–316). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

English, L., Watters, J. (2004). Mathematical Modelling in the Early School Years. *Mathematics Education Research Journal, 16*, 59-80.

English, L.D. (2003). Reconciling theory, research, and practice: A models and modelling perspective. *Educational Studies in Mathematics, 54*, 225–248.

Erickson, J., & Lehrer, R. (1998). The evolution of critical standards as students design hypermedia documents. *The Journal of the Learning Sciences, 7*(3-4), 351-386.

Freudenthal, H. (1971). Geometry between the devil and the deep sea. *Educational Studies in Mathematics*, *3*(3-4), 413-435.

Freudenthal, H. (1991). *Revisiting Mathematics Education.* China Lectures. Dordrecht: Kluwer Academic Publishers.

Goos, M. (1996). Making sense of mathematics: The teachers' role in establishing a classroom community of practice. Paper presented at *Annual Postgraduate Research Conference of the Graduate School of Education*, Queensland.

Gravemeijer, K. (1997). Commentary. Solving word problems: A case of modelling? *Learning and Instruction*, *7*(4), 389-397.

Gravemeijer, K., & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, *39*, 111-129.

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing, modelling and instructional design. In P. Cobb, E. Yackel, and K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 225-274). Mahwah, NJ: Lawrence Erlbaum Associates.

Greer, B. (1997). Modelling reality in mathematics classrooms: The case of word problems. *Learning and Instruction*, *7*(4), 293-307.

Haines, C., & Crouch, R. (2001). Recognizing constructs within mathematical modelling. *Teaching Mathematics and its Applications*, *20*(3), 129-138.

Haines, C., Izard, J., & Le Masurier, D. (1993). Modelling intentions realised: Assessing the full range of developed skills. In T. Breiteig, I. Huntley, and G. Kaiser-Messmer (Eds.), *Teaching and learning mathematics in context* (pp. 200-211). New York, NY: Ellis Horwood.

Harel, G., & Lesh, R. (2003). Local conceptual development of proof schemes in a cooperative learning setting. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: A models & modelling perspective on mathematics problem solving, learning & teaching* (pp. 359–382). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Hatano, G. (1997). Commentary. Cost and benefit of modelling activity. *Learning and Instruction*, *7*(4), 383-387.

Hiebert, J., Thomas P., Carpenter, E., Fennema, K., Fuson, P., Hanlie, M., Alwyn, O., & Wearne, D. (1996). Problem Solving as a Basis for Reform in Curriculum and Instruction: The Case of Mathematics, *Educational Researcher*, *25*(4), 12-21.

Hjalmarson (2005). Designing presentation tools: A window into teacher practice. *Unpublished Doctoral Dissertation.* Purdue University.

Houston, S. (1994). Comprehension tests in mathematics. *Teaching Mathematics and its Applications*, *12*(2), 60-73.

Houston, S., & Lazenbatt, A. (1999). Peer tutoring in a modelling course. *Innovations in Education and Training International*, *36*(1), 71-79.

Jones, G., Rich, B., & Day, R. (1996). The assessment of a complex problem: The bungee jump problem. *School Science and Mathematics*, *96*(2), 68-74.

Kaiser, Blomhøj & Sriraman (2006). Towards a didactic theory for mathematical modelling. *Zentralblatt für Didaktik der Mathematik*, 38(2), 82-85.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *Zentralblatt für Didaktik der Mathematik*, 38(3), 302-310.

Kelly, A. E., & Lesh, R. A. (Eds.) (2000). *Handbook of Research Design in Mathematics and Science Education*. Mahwah, NJ: Lawrence Erlbaum.

Kitchen, A. (1993). The 'Mechanics in Action Project'. In S. Houston (Ed.), *Developments in curriculum and assessment in mathematics*. (p. 57-66) Coleraine, Northern Ireland: University of Ulster.

Koellner Clark, K., & Lesh, R. (2003). A modelling approach to describe teacher knowledge. In R. Lesh and H. M. Doerr (Eds.), *Beyond constructivism: A models & modelling perspective on mathematics problem solving, learning & teaching* (pp. 159–173). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lehrer, R., & Schauble, L. (2003). Origins and evolution of model-based reasoning in mathematics and science. In R. Lesh and H. M. Doerr (Eds.), *Beyond constructivism: A models & modelling perspective on mathematics problem solving, learning & teaching* (pp. 59–70). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R., & Doerr, H. (1998). Symbolizing, communicating, and mathematizing: Key components of models and modelling. In P. Cobb and E. Yackel (Eds.), *Symbolizing, communicating, and mathematizing.* Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R., & Doerr, H. M. (2000). Symbolizing, communicating, and mathematizing: Key components of models and modelling. In P. Cobb, E. Yackel, and K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools and instructional design*. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R., & Doerr, H.M. (2003a). *Beyond Constructivism: A Models and Modelling Perspective on Mathematics Problem Solving, Learning and Teaching*, Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R., & Doerr, H.M. (2003b). In what ways does a models and modelling perspective move beyond constructivism? In R. Lesh and H.M. Doerr (Eds.), *Beyond constructivism: A models & modelling perspective on mathematics problem solving, learning & teaching* (pp. 383–403). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R., & Heger, M. (2001).Mathematical abilities that are most needed for success beyond school in a technology based age of information (invited). *The New Zealand Mathematics Magazine, 38*(2), 1-15.

Lesh,R., Kaput, J., & Hamilton, E. (Eds.) (2007, in press), *Foundations for the Future: The Need for New Mathematical Understandings & Abilities in the 21st Century*. Hillsdale, NJ: Lawrence Erlbaum Associates.

Lesh, R., & Lehrer, R. (2003). Models and Modelling Perspectives on the Development of Students and Teachers. *Mathematical Thinking and Learning, 5*(2-3), 109-130.

Lesh, R., Landau, M., & Hamilton, E. (1983). Conceptual models in applied mathematical problem solving. In R. Lesh (Ed.), *The acquisition of mathematical concepts and processes*. New York Academic Press.

Lesh, R. A. (2002). Research design in mathematics education: Focusing on design experiments. In L. D. English (Ed.), *Handbook of International Research in Mathematics Education: Directions for the 21st century* (pp. 27–50). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R. A., & Kelly, A .E. (2000). Multi-tiered teaching experiments. In R. A. Lesh and A. Kelly (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 197–231). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R. A., & Lehrer, R. (2003). Models and modelling perspectives on the development of students and teachers, *Mathematical Thinking and Learning 5*(2-3), 109–130.

Lesh, R., & Sriraman, B. (2005). Mathematics Education as a design science. *Zentralblatt für Didaktik der Mathematik* (International Reviews on Mathematical Education), 37(6), 490-505.

Lesh, R., & Yoon, C. (2004). Evolving Communities of mind- In which development involves several interacting and simultaneously developing strands. *Mathematical Thinking and Learning, 6*(2), 205-226.

Lesh, R., Cramer, K., Doerr, H. M., Post, T., & Zawojewski, J. (2003). Model development sequences. In R. Lesh and H. M. Doerr (Eds.), *Beyond constructivism: A models & modelling perspective on mathematics problem solving, learning & teaching* (pp. 35–58). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Lesh, R., Doerr, H. M., Carmona, G., & Hjalmarson, M. (2003).Beyond constructivism. *Mathematical Thinking and Learning, 5*(2), 211-234.

Lesh, R., Lester, F., & Hjalmarson, M. (2003). A models and modelling perspective on metacognitive functioning in everyday situations where mathematical constructs need to be developed. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: A models & modelling perspective on mathematics problem solving, learning & teaching* (pp. 383-404). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Lester, F. K. & Kehle, P. E. (2003). From problem solving to modeling: The evolution of thinking about research on complex mathematical activity. In R. Lesh & H. Doerr, H. (Eds.) *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning and Teaching* (pp. 501-518). Mahwah, NJ: Erlbaum.

Maull, W., & Berry, J. (2001). An investigation of student working styles in a mathematical modelling activity. *Teaching Mathematics and its Applications*, *20*(2), 78-88.

McNair, R. (2000). Life outside the mathematics classroom: Implications for mathematics teaching reform. *Urban Education*, *34*(5), 550-570.

Mevarech, Z., & Stern, E. (1997). Interaction between knowledge and contexts on understanding abstract mathematical concepts. *Journal of Experimental Child Psychology*, *65*(1), 68-95.

Money, R., & Stephens, M. (1993). A meaningful grading scheme for assessing extended tasks. In N. Webb and A. Coxford (Eds.), *Assessment in the mathematics classroom. 1993 Yearbook.* (Ch. 21) Reston, VA: NCTM.

Mousoulides, N., Pittalis, M., & Christou, C. (2006). Improving Mathematical Knowledge through Modelling in Elementary School. Paper accepted for publication in the 30th *Conf. of the Int. Group for the Psychology of Mathematics Education*. Prague.

Mullis, Martin, Gonzalez, & Chrostowski, 2004). *TIMSS 2003 International mathematics report.* Boston, MA: International Association for the Evaluation of Educational Achievement.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, and B. Findell (Eds). Mathematics Learning Subcommittee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Niss, M. (1987). Applications and modelling in the mathematics curriculum – State and trends. *International Journal of Mathematical Education in Science and Technology*, *18*(4), 487-505.

Niss, M. (1993). Assessment of mathematical applications and modelling in mathematics teaching. In J. de Lange, C. Keitel, I. Huntley and M. Niss (Eds.), *Innovation in Mathematics Education by Modelling and Applications*. (p. 41-51) Chichester: Ellis Horwood.

Organization for Economic Co-Operation and Development [OECD] (2004). *Problem Solving for Tomorrow's World – First Measures of Cross Curricular Competencies from PISA 2003,* http://www.pisa.oecd.org/dataoecd/25/12/34009000.pdf. Retrieved 29.09.2005.

Pace, S. (2000). Teaching mathematical modelling in a design contest: A methodology based on the mechanical analysis of a domestic crusher. *Teaching Mathematics and its Applications*, *19*(4), 158-165.

Polya, G. (1962). *Mathematical discovery*. New York: Wiley.

Polya, G. (1973). *How to solve it: A new aspect of mathematical methods*. (2nd Ed.). Princeton, NJ: Princeton University Press.

Resnick, L. (1988) Treating mathematics as an ill-structured discipline. In R. Charles and E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 32-60). Reston, VA: NCTM.

Reusser, K., & Stebler, R. (1997). Every word problem has a solution - The social rationality of mathematical modelling in schools. *Learning and Instruction*, *7*(4), 309-327.

Roth, W., & McGinn, M. (1997). Toward a new perspective on problem solving. *Canadian Journal of Education*, *22*(1), 18-32.

Sauian, M. (2002). Mathematics education: The relevance of "contextual teaching" in developing countries. In P. Valero and O. Skoysmose (Eds.), *Proceedings of the 3rd International MES Conference*. (Ch. 1) Copenhagen: Centre for Research in Learning Mathematics.

Schoenfeld, A. (1982). Some thoughts on problem-solving research and mathematics education. In F. K. Lester, Jr., and J. Garofalo (Eds.), *Mathematical Problem Solving: Issues and Research* (pp. 27-37). Philadelphia, PA: The Franklin Institute Press.

Schoenfeld, A. (1987). Confessions of an accidental theorist. *For the Learning of*

Mathematics, *7*(1), 30-38.

Schoenfeld, A. (1989). Problem solving in context(s). In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (p. 82-92). Reston, VA: NCTM.

Schoenfeld, A. (1991). On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J. Voss, D. Perkins and J. Segal (Eds.), *Informal reasoning and education* (p. 311-343). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. (Ch. 15) New York: Simon & Schuster Macmillan.

Schoenfeld, A. (1994). Reflections on doing and teaching mathematics. In A. Schoenfeld (Ed.), *Mathematical thinking and problem solving*. (Ch. 3) Hillsdale, NJ: Lawrence Erlbaum Associates.

Schorr, R., & Koellner-Clark, K. (2003). Using a Modelling Approach to Analyze the Ways in Which Teachers Consider New Ways to Teach Mathematics. *Mathematical Thinking and Learning, 5*(2-3), 191-210.

Schorr, R., & Lesh, R. (2003). A modelling approach for providing teacher development. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modelling perspectives on mathematics problem solving, learning, and teaching* (pp. 141-158). Mahwah, NJ: Lawrence Erlbaum Associates.

Sfard, A.(1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational Studies in Mathematics, 22*, 1-36.

Smith, R., & Thatcher, D. (1989). An examination for mathematical modelling. *International Journal of Mathematical Education in Science and Technology*, *20*(4), 605-613.

Sriraman, B., & Lesh, R. (2006). Beyond Traditional conceptions of modelling. *Zentralblatt fuer Didaktik der Mathematik*, *38*(3), 247-254.

Stanic, G., & Kilpatrick, J. (1987). Historical perspectives on problem solving in the mathematics curriculum. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving*. (Ch. 1) Reston, VA: NCTM

Staub, F., & Reusser, K. (1995). The role of presentational structures in understanding and solving mathematical word problems. In C. A. Weaver, S. Mannes, and C. R. Fletcher (Eds.), *Discourse comprehension: Essays in honor of Walter Kintsch* (p. 285-305). Hillsdale, NJ: Lawrence Erlbaum Publishing.

Stillman, G. (2001). Impact of school-based assessment on the implementation of a modelling/applications-based curriculum: an Australian example. *Teaching Mathematics and its Applications*, *20*(3), 101-107.

Verschaffel, L., & De Corte, E. (1997). Teaching realistic mathematical modelling in the elementary school: A teaching experiment with fifth graders. *Journal for Research in Mathematics Education*, *28*(5), 577-601.

Verschaffel, L., De Corte, E., & Borghart, I. (1997). Pre-service teachers' conceptions and beliefs about the role of real-world knowledge in mathematical modelling of school word problems. *Learning and Instruction*, 7(4), 339-359.

Verschaffel, L., De Corte, E., & Lasure, S. (1994). Realistic considerations in mathematical modelling of school arithmetic word problems. *Learning and Instruction*, *4*, 273-294.

Webb, N. (1992). Assessment of students' knowledge of mathematics: Steps toward a theory. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*, New York: Macmillan.

Wyndhamn, J., & Säljö, R. (1997). Word problems and mathematical reasoning – A study of children's mastery of reference and meaning in textual realities. *Learning and Instruction*, *7*(4), 361-382.

Yoshida, H., Verschaffel, L., & De Corte, E. (1997). Realistic considerations in solving problematic word problems: Do Japanese and Belgian children have the same difficulties? *Learning and Instruction*, *7*(4), 329-338.

Zawojewski, J. S., & Lesh, R. (2003). A models and modelling perspective on problem solving. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modelling perspectives on mathematics problem solving, learning, and teaching* (pp. 317-336). Mahwah, NJ: Lawrence Erlbaum Associates.

Zawojewski, J. S., Lesh, R., & English, L. (2003). A models and modelling perspective on the role of small group learning activities. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modelling perspectives on mathematics problem solving, learning, and teaching* (pp. 337-358). Mahwah, NJ: Lawrence Erlbaum Associates.