Commentary on Chapter 11

On Proof and Certainty- Some Educational Implications

Bharath Sriraman
The University of Montana

Hillary VanSpronsen
Michigan Technological University

Nick Haverhals
The University of Montana

In the domain of proofs in mathematics education, the DNR[1] theory created by Guershon Harel attempts to bridge the epistemologies of what constitutes proofs in professional mathematics and mathematics education. The DNR theory is one that has been gradually developed. In one of the "early" papers that proposed this theory, Harel (2006a) wrote:

> *Pedagogically, the most critical question is how to achieve such a vital goal as helping students construct desirable ways of understanding and ways of thinking. DNR has been developed to achieve this very goal. As such, it is rooted in a perspective that positions the mathematical integrity of the content taught and the intellectual need of the student at the center of the instructional effort. The mathematical integrity of a curricular content is determined by the ways of understanding and ways of thinking that have evolved in many centuries of mathematical practice and continue to be the ground for scientific advances. To address the need of the student as a learner, a subjective approach to knowledge is necessary. For example, the definitions of the process of "proving" and "proof scheme" are deliberately student-centered. It is so because the construction of new knowledge does not take place in a vacuum but is shaped by one's current knowledge.(p.23, pre-print)*

Harel's views in a sense echo the recommendations of William Thurston, the 1982 Fields medal winner, whose article *On Proof and Progress* (see Hersh, 2006) gets widely cited and reprinted in both the mathematics and the mathematics education communities. Thurston outlines for the lay person:
(1) what mathematicians do
(2) how (different) people understand mathematics
(3) how this understanding is communicated,
(4) what is a proof
(5) what motivates mathematicians, and finally
(6) some personal experiences.

Thurston stresses the human dimension of what it means to do and communicate mathematics. He also gives numerous insights into the psychology of mathematical creativity, particularly in

---

[1] DNR= *duality*, *necessity*, and *repeated-reasoning*

the section on what motivates mathematicians. Mathematics educators can draw great satisfaction from Thurston's writings, particularly on the need for a community and communication to successfully advance ideas and the very social and variant nature of proof, which depends on the sophistication of a particular audience. It seems that although there are dissonances in the terminology used by psychologists, mathematics educators, and mathematicians when speaking about the same construct, there are some similar elements which can lay the foundation of a common epistemology (Törner & Sriraman, 2007). But several hurdles exist in creating common epistemologies for the diverse audience of researchers working in mathematics education. For instance Harel (2006b) in his commentary to Lester's (2005) recommendations to the mathematics education research community (for developing a philosophical and theoretical foundation), warns us of the dangers of oversimplifying constructs that on the surface seem to be the same. The original commentary to Lester appears in this volume. Harel (2006b) also wrote that a major effort has been underway for the last two decades to promote argumentation, debate and discourse in the mathematics classroom. He points out that scholars from multiple domains of research have been involved in this initiative, i.e., mathematicians, sociologists, psychologists, classroom teachers, and mathematics educators. In his words:

However, there is a major gap between "argumentation" and "mathematical reasoning" that, if not understood, could lead us to advance mostly argumentation skills and little or no mathematical reasoning. *Any research framework for a study involving mathematical discourse ...[w]ould have to explore the fundamental differences between argumentation and mathematical reasoning, and any such exploration will reveal the critical need for deep mathematical knowledge. In mathematical deduction one must distinguish between status and content of a proposition (see Duval, 2002). Status (e.g., hypothesis, conclusion, etc.), in contrast to content, is dependent only on the organization of deduction and organization of knowledge. Hence, the validity of a proposition in mathematics—unlike in any other field—can be determined only by its place in logical value, not by epistemic value (degree of trust).*

However, the community of mathematicians has on numerous occasions placed epistemic value on results before they completely agree logically with other related results that lend credence to its logical value. Historical examples that convey the interplay between the logical and the epistemic can be seen in the (eventual) marriage between non-Euclidean geometries and modern Physics. If one considers Weyl's (1918) mathematical formulation of the general theory of relativity by using the parallel displacement of vectors to derive the Riemann tensor, one observes the interplay between the experimental (inductive) and the deductive (the constructed object). The continued evolution of the notion of tensors in physics/Riemmanian geometry can be viewed as a culmination or a result of the flaws discovered in Euclidean geometry. Although the sheer beauty of the general theory was tarnished by the numerous refutations that arose when the general theory was proposed, one cannot deny the present day value of the mathematics resulting from the interplay of the inductive and the deductive. According to Bailey & Borwein (2001), Gauss used to say *I have the result but I do not yet know how to get it* He also considered that, to obtain the result, a period of *systematic experimentation* was necessary. There is no doubt then, that Gauss made a clear distinction between *mathematical experiment* and *proof*. In fact, as Gauss expressed, we can reach a level of high certitude concerning a mathematical fact before the proof, and at that moment we can decide to look for a proof. Many of Euler's results on infinite series have been proven correct according to modern standards of rigor. Yet, they were already established as valid results in Euler's work. Then, what has remained and what has changed in these theorems? If instead of looking at foundations we choose to look at mathematical results, as resulting from a human activity that is increasingly refined, then we could find a way to answer that difficult question. This perspective coheres with the view that mathematical ideas can be thought through successive levels of formalizations. *The theorem is the embodied idea*: the proof reflects the level of understanding of successive generations of mathematicians. Different proofs of a theorem cast light on different faces of the embodied idea (Moreno & Sriraman, 2005).

V.I. Arnold (2000) one of the most distinguished mathematicians of the last decades, has said:

> *Proofs are to mathematics what spelling (or even calligraphy) is to poetry. Mathematical works consists of proofs as poems consist of characters.*

In the same paper, Arnold (2000) quotes Sylvester saying that:

> *A mathematical idea should not be petrified in a formalised axiomatic setting, but should be considered instead as flowing as a river*. (p. 404)

There are a number of approaches to the teaching and learning proof. The DNR approach can be compared and contrasted with two earlier approaches namely the deductive approach (e.g., Fawcett (1938/1966) and the heuristic approach (e.g., Polya, 1954).

In the DNR approach, mathematics is up broken into two categories: "ways of thinking" (the subject matter at hand and ways the subject matter is communicated) and "ways of understanding" (the way that one approaches and/or views subject matter). Labeling these as separate constructs gives a tool for considering mathematics education. As things currently stand, mathematics education is primarily concerned with ways of understanding. Educators first consider the material that is to be taught and how it fits together logically (in relation to itself or later material to be taught). After this examination, the material is presented to the students in a manner matching the logical construction of the material. What is missing, then, is the consideration for the students' ways of thinking. This attention to ways of understanding has consequences beyond how curriculum presented to students is composed. It also affects how the mathematics pre-service teachers are taught and causes teachers to lose sight of aspects of students' cognitive development, e.g. students are taught definitions without efforts to develop *definitional reasoning*. The focus on ways of understanding leads to curricular development as mere content sequencing, with "no or scant attention to … the complexity of the process involved in acquiring and internalizing" the content (Harel, 2008, p 495). To be better at teaching and learning proof, attention needs to be paid to not only how results fit together, but also to how they are perceived by the students.

One could make the case that the deductivist approach used by Fawcett (1938) gives consideration to both ways of thinking and ways of understanding. In the experimental classroom set up in the classic book *The Nature of Proof*, students are guided to certain theorems (by a teacher who gave consideration to ways of understanding). However, the way in which the students come to the theorems and their proofs is collaborative and student driven (more on this later). As such, ways of thinking are taken into account naturally. More than this, the class is specifically designed to appeal to students' ways of thinking. No textbooks are used other than the ones students create themselves. Although ways of thinking are taken into account, ways of understanding play a large role in the class. While the proofs themselves are student created, the format they take on is largely dictated by the teacher. The first objective of the class is to emphasize the importance of definitions and accepted rules. The class is also trained in identifying hidden assumptions and terms that need no definition. This is meant to train students in deductive reasoning. That is, students are trained to start with agreed upon premises (be they axioms, definitions, or generally accepted criteria outside of mathematics) and produce steps that lead to the sought conclusions. Included in this training is the analysis of other arguments on the basis of how well they do the same. This is in conflict with the DNR approach in that it allows for only a single method of proof. Direct proof is given to students with little regard to the way in which they will internalize the method. If a particular student is creative in his or her mathematical thought, it ought to be appealed to in the teaching of proof. In the book *Proofs and Refutations* (1976), Lakatos makes the point that this sort of "Euclidean methodology" is detrimental to the exploratory spirit of mathematics. Not only can an over-reliance on deduction dampen the discovery aspect of mathematics, it can also ignore the needs of students as they learn proof.

While the DNR approach would insist the teaching and learning of proof take into account the students' perspectives and the deductive approach would have students learn how to become proficient at direct proofs, the heuristic approach to teaching and learning proof would keep an eye on the ways that mathematicians work. In *Patterns of Plausible Inference,* Polya (1954), lays out the ways in which people judge the plausibility of statements. By doing so, he gives a guide for students as they go about exploring the validity of a statement. "I address myself to teachers of mathematics of all grades and say: *Let us teach guessing!*" (Polya, 1954, p.158) This is quite different from the deductive view which holds fast to inferences that can be logically concluded, where inconclusive but suggestive evidence has no place. While we do not doubt that the deductivist approach leaves room for guessing, it is not its primary emphasis. This is not to say, either, that the heuristic approach would abandon demonstrative proof. However, it is similar to the DNR perspective in that it would add to it. Where the DNR approach would give consideration to students' perspectives on proof, the heuristic approach would try to help shape it.

As mentioned above, in the heuristic approach students would be taught to act in ways similar to mathematicians when they are judging the potential validity of a statement and looking for proof. Lakatos (1976) makes a similar case. In his fictional class, the students argue in a manner that mirrors the argument the mathematical community had when considering Euler's formula for polyhedra. He states that an overly deductive approach misrepresents the ways the mathematics community really works. Fawcett shows, however, a way in which a deductivist classroom can model the mathematical community to a certain extent. Like in the mathematics community, disagreements arise and the need for convincing fosters the need for proof. This is also similar to the heuristic approach in that non-conclusive evidence can be considered that can help sway opinions. This evidence can also lead to improved guesses and more efficient proof attempts, as Polya shows. Thus, the DNR approach to proof has not come "out of the blue" so to speak but is anchored in previous elements in the literature.

The framework of DNR-based instruction is based on eight initial premises. The first premise, Mathematics, ties into the differences between the process and the product of proof writing. That is, the process in which students engage during the stages of proof writing may include ideas, investigations, conjectures, trials, and arguments that are not included or in evidence in the final product, i.e. the proof itself. Therefore, the learning of mathematical proof writing should include discussion of the products (ways of understanding) and also of the processes used to create this product (ways of thinking).

The second category of premises, Learning, mirrors the process of proving in the sense that proving can be thought of as convincing oneself, convincing a friend, and convincing an enemy (Mason, Burton, and Stacey, 1982). Convincing oneself, and the desire to do so, could be considered as the idea of *epistemophilia*, or the love of knowledge. We see numerous times that a student is motivated by the need to prove something to be true first to themselves. This is a desire to investigate a theorem for purely knowledge sake. Motivation for the final two phases, however, is sometimes lacking, which we will discuss in regards to necessity later. The second phase of proof writing, convincing a friend, means that one must find a way to communicate his or her understanding of a proof to another individual, in essence truly *knowing* the theorem. Lastly, a proof needs to be solid enough to convince even an enemy of its truth. Students who reach this point will truly own a proof in their own knowledge, perhaps even further convinced of its truth through a process of investigating, posing, and proving their own theorems, in other words making the connection between *knowing* and *knowledge* of others' proofs.

All of this learning of proof is not done out of context. Leaving aside for the moment the issue of whether to introduce proof writing in a specific course intended for this purpose or in direct context during other courses, we instead focus on the fact that in either learning environment, content and context should be brought in to relate proof writing to specific mathematics. Without this *context dependency,* we would essentially be teaching logic, argumentation, and justification in the general sense with no real implication for how mathematics is dependent upon proof writing, or of the specific tools used in mathematical proofs. It is also important to note that, specific to mathematics, there is a need to have content knowledge for the ability to be successful in proof writing. In a study of 40 high school and 13 college students, Baker (1996) found that "[a] primary source of difficulty was attributable to a lack of mathematical content knowledge" (p. 15). Therefore, context dependency refers to the connection between proof-writing knowledge and knowledge within the content domain.

There is also a connection between proof writing and the premises laid out by Harel in the third category, Teaching. Particularly during the initial stages of learning to write proofs, students can progress further with guided assistance from instructors and knowledgeable peers than they can alone. This should come as no surprise in mathematics, as collaboration is encouraged among colleagues throughout academia even for those with significant experience. Working with others, especially those with more training and background in proof writing, can give the push that is needed to work past the "stuck" points in a proof and get thoughts back on track towards the goal. Mingus and Grassl (1999) studied the beliefs and experiences of pre-service teachers in mathematics. One student in this study said, "I struggled with the proof process and as a result, we did all of our proofs in groups of two or more students" (p. 440). Others here share the workload and offer a different perspective, which also allows for possible growth in the ways of understanding and thinking that accompany proof writing.

We have already hinted at the ways in which we believe the Concepts of DNR, i.e. ways of understanding and ways of thinking, relate to proofs. Specifically, ways of understanding include the proof as a product and ways of thinking include the processes involved in constructing proofs. Ways of thinking can also be described as the overarching beliefs that affect the ways in which we choose the cognitive tasks involved in proof writing. Harel refers to these processes as proof schemes, which can be categorized as external conviction, empirical, or deduction (Harel and Sowder, 1998). These categories point to the underlying beliefs that we have about what constitutes a valid proof. Weber (2004) categorizes proof attempts into three similar categories: procedural, syntactic, and semantic. Four levels of proof were identified by Balachef (1988): naïve empiricism, crucial experiment, generic example, and thought experiment. All of these involve the beliefs of what constitutes proof, and the related link between these beliefs and chosen cognitive tasks. Van Dormolen (1977) classified levels of thinking during proof writing according to the van Hiele levels of development in geometric thought: ground level, and the first and second levels of thinking. We can see from these examples over multiple decades of mathematics education research that the study of ways of thinking in mathematics, and in particular in proof writing, is not a new concept.

The relationship between DNR-related instruction and proof writing instruction are found in the instructional principles outlined by Harel that define the acronym used for this theory. *Duality* tells us that our beliefs about proof influence the proofs that we construct, and that the proofs we construct also influence our beliefs about proof. Evidence has shown that this portion of the theory holds true. For example, even after specific training, students do not always understand that evidence is not proof (Chazan, 1993). Similar results have shown that the distinction between inductive and deductive reasoning is not clear to our students (c.f. Weber, 2003; Coe and Ruthven, 1994; Martin and Harel, 1989; and Williams, 1980).

When studying the proof-writing strategies of college students writing mathematical proofs, VanSpronsen (2008) found that several students used similar strategies in each proof attempt, regardless of the theorem posed, or without instantiating the new methods recently learned. That is, new methods were introduced to students, but they failed to adopt these methods when not under the supervision of an instructor or other classmates. Instead, several had difficulty moving past their desire for a proof in a certain format that was deemed as the "best" or "most valid" type of proof in their mind (e.g. proofs involving equation manipulation). Baker (1996) found that students focused more on the form of the proof than the accuracy or substance, indicating that "their cognitive attention [was focused] on procedures rather than on concepts or applications" (p. 13). Students also rely on memorization of previous proofs to create or verify new proofs without an understanding of the underlying mathematics (Weber, 2003). Moore (1994) found, in a study involving 16 students in a transition-to-proof undergraduate course, that "several students in the transition course had previously taken upper-level courses requiring proofs. All of them said they had relied on memorizing proofs because they had not understood what a proof is nor how to write one" (p. 264).

We are not without hope, however, of changing these patterns, as the dual nature of this theory implies we can change the ways of understanding by opening up new ways of thinking over time and with motivation to do so. The motivational aspect of proof writing, that is the need or desire to actually prove a statement in a formal way, is the portion of this process that is defined as *Necessity* in the DNR theory. It should not be surprising to anyone who has taught beginning mathematics proof writers that students are often unmotivated to follow through on a proof. Almeida (2001) found that high school participants in his study from the UK were able to form conjectures, but “were not motivated to explain or justify them until the interviewer teased out their often original arguments” (p. 59).

Another example of this apparent lack of motivation comes when we ask our students to prove a fact that seems obviously true. Students are heard saying things like, “Isn’t that just always true?” or “I have assumed that in all my math classes before, why do I have to prove it now?”, showing a clear desire for a reason, or motivation, to actually complete the proof. While the DNR theory does not yet offer any pedagogical methods for dealing with these questions, it does raise an awareness of the issue and points out the need to address this when teaching students mathematics and, in this case, teaching students to write mathematical proofs.

The last instructional principle, *Repeated Reasoning*, is in evidence as we watch students mature into expert proof writers. This does not happen immediately in any one course, but rather is a process that occurs over time during many courses. Students even agree with this principle, and repeated exposure to proof throughout different courses gives students confidence in proof writing (Mingus and Grassl, 1999). This repetition of ideas, motivation, and proof theory allows students to internalize new ways of thinking and changes their way of understanding in turn. Again, this does not speak to the ongoing discussion in mathematics education of when and how proof writing should be introduced (in one specific course or within the content of several courses), but rather points out that regardless of which choice is made expert proof writing should not be expected instantaneously. Instead, we should strive to make consistent repeated exposures to these ideas throughout our sequence of course offerings to follow through on whichever initial exposure students have experienced.

All three of these principles (duality, necessity, and repeated reasoning) are interrelated throughout mathematics proof writing. As the literature suggests, none are new concepts in the research. However, to bring them all together into one theory, as one conceptual framework together, is a new and intriguing idea. Rather than treat any one component separately from the others, DNR-based instruction suggests that we attend to all three simultaneously, and take care to employ ideas from these areas in our own teaching and when designing and conducting research in mathematics education.

As with any theory, DNR has potential issues as well. Duality theory is tricky ground to tread in education in general, as the distinction between ways of understanding and ways of thinking is difficult to define clearly and even harder to analyze within students. Particularly, ways of thinking are not readily available for data collection in any one episode of proof writing, but rather must be characterized over several attempts over time. The interpretation of metacognitive behaviors, or thoughts about one’s thinking, is not transparent or without issues of reliability and validity in the collection of data. Therefore, the ability to collect data on the relationships between ways of understanding and ways of thinking requires forethought and a strong additional framework outside of that presented in the chapter.

One point raised by Harel was that often mathematics education research is not at all mathematically focused, “one is left with the impression that the report would remain intact if each mention of ‘mathematics’ in it is replaced by a corresponding mention of a different academic subject” (p. 1). However, this is an issue that Harel himself did not directly address in regards to his own theory of DNR in this chapter. While Harel did include Mathematics as a premise under which this theory was developed, and he did give specific examples of the theory and the lesson that were mathematically rich in content, the underlying question of whether this could be applied outside of mathematics education still remains. The Mathematics premise is the only premise that appears to be unique to mathematics. All others could be applied to various areas of education with little loss of meaning or applicability. How then is DNR-based instruction math specific? What portions of DNR are not directly applicable in other areas and are unique to mathematics? Perhaps these ideas are underlying Harel's design, and we believe that in light of his own comments this framework was developed with the intent to be math specific, however Harel does not address this issue in the chapter.

Even proponents of this theory may ask how DNR-based instruction could be implemented in the classroom, especially after Harel himself stated that this is an area yet to be tapped within this framework. In 2003, Harel conducted a study in which the effect of DNR-based instruction was investigated in relation to the teaching practices of in-service teachers. While the results support the framework developed in this chapter, in the end we are left to wonder how we can successfully implement these ideas if "even intensive professional development spanning a two-year period [was] not sufficient to prepare teachers to be autonomous in altering their current curricula to be consistent with *DNR*" (p. 3). How great a time commitment must we then invest to affect the change desired through this framework? How can we be encouraged to further research these ideas if implementation seems nearly impossible? We are reminded of the many promising results in the problem-solving literature that arose in the 1980s, only to learn later that implementing change in the teaching and learning of problem solving was a much greater task than originally anticipated (c.f. Schoenfeld, 1992).

There is an additional issue in implementation, even if we disregard the actual training of teachers to successfully achieve the goals set out by DNR, of the apparent extra time necessary to properly allow students to work through the process of self-motivation, discovery, and attending to the intellectual and instructional needs and principles described by Harel. In the ever-shrinking time allotted in our classes, both at the secondary and post-secondary levels, with increasing demands on the content to be addressed, is this time demand feasible? Can we achieve these seemingly important goals of duality, necessity, and repeated reasoning, and also the content specific goals in our areas? Until further research gives us an indication of what a typical classroom using DNR would look like, it is impossible to say whether the time needed for this inquiry-based, discovery-based learning is worth the reward.

There are other issues that will not be answered until instructional methodologies are designed, implemented, and studied, such as how much repetition is necessary to achieve our goals, how we move our students through this process, and whether these methods address multiple learning styles. These are not flaws in the framework, necessarily, but rather things to consider while moving forward within the framework.

All this being said, both positive and negative views in focus, we believe that DNR-based instruction does present a new conceptual framework of value in mathematics education research, particularly in the domain of proof writing. There are larger issues to consider that are not yet resolved, however the ability of this theory to bring together the major ideas related to instruction in mathematics proof writing give it the potential to be a strong backing for future research and a reasonable theory upon which to construct new teaching tools and research designs.

References

Arnold, V.I. (2000) *Mathematics: Frontiers and Perspectives.*American Mathematical Society.

Almeida, D. (2001). Pupils' proof potential. *International Journal of Mathematical Education in Science and Technology, 32*(1), 53-60.

Baker, J. D. (1996). *Students' difficulties with proof by mathematical induction*. Paper presented at the Paper presented at the Annual meeting of the American Educational Research Association.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, Teachers and Children* (pp. 216-236). Great Britain: Hodder and Stoughton Educational.

Bailey, D. & Borwein, P. (2001), *Mathematics Unlimited 2001and Beyond*. B. Engquist and W. Schmid (eds), Springer-Verlag.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence nad mathematical proof. *Educational Studies in Mathematics,* 24 (4), 359-387

Coe, R., & Ruthven, K. (1994). Proof practices and constructs of advanced mathematical students. *British Educational Research Journal, 20*(1), 41-54.

Duval, R. (2002). Proof understanding in mathematics. *Proceedings of 2002 International Conference on Mathematics: Understanding Proving and Proving to Understand* (pp. 23-44). Taiwan: National Taiwan Normal University, Department of Mathematics.

Fawcett, H. P. (1938/1966). *The nature of proof: A description and evaluation of certain procedures used in senior high school to develop an understanding of the nature of proof.* New York, NY: AMS Reprint Company.

Harel, G. (2006a). What is mathematics? A pedagogical answer to a philosophical question. In R.B. Gold & R. Simons (Eds.), *Current Issues in the Philosophy of Mathematics From the Perspective of Mathematicians*. Mathematical Association of America. Pre-print accessed at http://math.ucsd.edu/~harel/downloadablepapers/Harel%27s%20WhatIsMathematics.pdf

Harel, G. (2006b). Mathematics education research, its nature, and its purpose: a discussion of Lester's paper. *Zentralblatt für Didaktik der Mathematik (ZDM)*, 38(1), 58-62.

Hersh, R. (2006). *18 Unconventional Essays on the nature of mathematics*. Springer: New York

Harel, G. (2008). *DNR* perspective on mathematics curriculum and instruction, Part I: focus on proving. *ZDM-The International Journal on Mathematics Education*, 40 (3), 487-500.

Lakatos, I. (1976). *Proofs and Refutations: The Logic of Mathematical Discovery*. Cambridge: Cambridge University Press.Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*(1), 41-51.

Mason, J., Burton, L., & Stacey, K. (1984). *Thinking Mathematically*. London: Addison-Wesley Publishing Co.

Mingus, T. T. Y., & Grassl, R. M. (1999). Preservice teacher beliefs about proofs. *School Science and Mathematics, 99*(8), 438-444.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*(3), 249-266.

Moreno, L., & Sriraman, B. (2005). Structural stability and dynamic geometry: Some ideas on situated proofs. *ZDM-The International Journal on Mathematics Education, 37(3*), 130-139.

Polya, G. (1954). *Patterns of plausible inference*. Princeton, NJ: Princeton University Press.

Schoenfeld, A. (1992). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics. In D. Grouws (Ed.), *Handbook of research of teaching and learning* (pp. 334-370). New York: Macmillan Publishing Co.

Sowder, L., & Harel, G. (1998). Types of students' justifications. *The Mathematics Teacher, 91*(8), 670-675.

Törner, G., & Sriraman, B. (2007). A Contemporary Analysis of the six "Theories of Mathematics Education" Theses of Hans-Georg Steiner, *ZDM- The International Journal on Mathematics Education*, 39(1&2), 155-163.

Van Dormolen, J. (1977). Learning to understand what giving a proof really means. *Educational Studies in Mathematics, 8*, 27-34.

VanSpronsen, H. (2008). Proof processes of novice mathematics proof writers. (Doctoral Dissertation, The University of Montana, 2008). *Dissertation Abstracts International, 69, 04A (UMI No.* 3307220).

Weber, K. (2003). *A procedural route toward understanding the concept of proof.* Paper presented at the 27th Conference of the International Group for the Psychology of Mathematics Education, Honolulu, USA.

Weber, K. (2004b). A framework for describing the processes that undergraduates use to construct proofs. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, 4*, 425-432.

Weyl, H. (1918). *Raum-Zeit-Materie. Vorlesungen über allgemeine Relativitätstheorie*, Berlin: Springer Verlag.

Williams, E. (1980). An investigation of senior high school students' understanding of the nature of proof. *Journal for Research in Mathematics Education*, 165-166.