INVARIENCE OF RATIO:
THE CASE OF CHILDREN’S ANTICIPATORY
SCHEME FOR CONSTANCY OF TASTE

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In this article we study the concept of invariance of ratio through an investigation of children’s understanding of constancy of taste—that is, the notion that random samples of a given mixture taste the same—using a device that does not resort to conventional symbolism. The paper begins with a definition of constancy of taste and other quantitative analogues. Then it presents a theoretical analysis of how constancy of taste may emerge from the child’s additive world and grow into a conception where taste becomes an intensive quantity. The analysis suggests that one’s conception of taste constancy is linked in a fundamental way to one’s conception of invariance of ratio. Following this analysis, the paper reports a study that demonstrates the absence of taste constancy among sixth-grade children. More specifically, the study shows that sixth-grade children base their judgment of the relative strength of the taste of two samples from the same mixture on at least one of three (extraneous) variables: the relative volumes of the samples to be tasted, whether the mixture is thought of as consisting of a single ingredient or more than one ingredient, and the relative amount of the ingredients stated in the problem.

Although the proposition “two samples of the same homogeneous mixture do not necessarily taste the same” sounds illogical to us, statements to this effect were made by some of the children who participated in the study reported in this paper, without raising any conflict for them. What is the conceptual basis for the reasoning of these children? Our hypothesis is that they have not yet fully acquired the notion of constancy of taste. Constancy of taste is the assumption a person makes that, under ideal conditions, random samples of a given mixture taste the same, independent of the size or the spot in the mixture from which they are taken. A conception of taste constancy, we argue, underlies one’s conception of the quantity of taste as an intensive quantity (i.e., a quantity that emerges from a multiplicative

\[\text{1Subjectively speaking, that is, conditions that the person conceives as necessary for the assumption to be true.}\]

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comparison between two quantities) and as a ratio that is invariant under variations of the volume of a composite amount. Although research that used "mixture problems" to investigate the concept of proportion has never looked at children’s acquisitions of taste constancy or perhaps has taken it for granted, we report a study that demonstrates its absence with sixth-grade children.

The definition of taste constancy above is generalizable to other intensive quantities. For example, constancy of speed is the assumption a person makes that under ideal conditions the fastness of an object traveling along a given path is the same on any random interval of the path, independent of the length or the location of the interval. As with taste, we have witnessed children making statements that indicate they have not acquired speed constancy. For example, some children made statements to the effect that "two airplanes starting simultaneously from the same location and flying at constant speeds in a parallel path reach Town A at the same time, but one gets to Town B first." Taste and speed constancies are about the qualities of the senses of "taste" and "fastness." In general, we may talk about constancy of quantity's quality. We distinguish between constancy with respect to location and constancy with respect to measure. For example, constancy of taste with respect to location is the conception that the taste of a sample of a mixture is not dependent on the spot in the mixture from which the sample is taken, whereas constancy of taste with respect to measure is the conception that the size of the sample does not influence its taste. The study reported in this paper was directed to look at constancy of taste with respect to measure, not location.

CONSTANCY OF TASTE AND INVARIANCE OF RATIO:
A THEORETICAL ANALYSIS

In this section, we propose a hypothetical analysis of the acquisition of constancy of taste, how it may emerge from the child’s additive world, and how it may grow into a multiplicative concept where taste becomes an intensive quantity. Although the analysis deals specifically with the quantity of taste per se, it points to possible basic layers in the foundations for conceptions of intensive quantities in general. The study reported later in this paper was not intended to examine this analysis but was inspired by it.

The notion of "taste" at the empirical level is an action of actual experience, "the sense by which the flavor or savor of things is perceived when they are brought into contact with the tongue" (The Random House Dictionary,
At the operative level, "taste" is an anticipatory scheme, "a scheme that suggests an answer before the details are 'filled in' by the action during the actual process of arriving at it" (Piaget & Inhelder, 1967; p. 133). With this scheme, the child recognizes that tastes of samples of the same or different mixtures can be compared without carrying out the actual action of tasting the samples. The critical stage that leads to the construction of "taste" as an anticipatory scheme is when the child begins to realize that different samples of the same mixture taste the same even when they are of different sizes. This realization is critical because it is in conflict with the child's current quantitative world, and it is from this conflict that taste constancy emerges. The conflict encountered is this: At an early age, when such a realization is likely to be made, the child's most predominant conceptual field is additive in nature; that is, quantitative operations available to the child at this age are predominantly additive. In this world, the child's experience—and so her or his expectation—is that quantitative operations always result in a change in the measure of the qualities of the operands (e.g., when two collections are put together, the measure of their "muchness" increases). This child's additive world is in conflict with her or his experience with taste because when the size of a mixture's sample is varied (made greater or smaller—definitely an additive operation) the taste of the samples (i.e., the measure of the quality of the operand) stays the same, which is against the child's expectation.

The natural desire for regulation would lead the child to seek a resolution of this conflict by theorizing what might be happening behind the scene, namely, there must be "something" that balances the constituent elements of the mixture so that the mixture's taste is invariant under a change of the sample's size. The theory constructed by the child involves two interrelated conceptions. The first is about the physical interaction between the constituent elements, where the child begins to discover that liquids, when mixed together, diffuse among each other uniformly (the Uniform Diffusion Principle, see Harel, Behr, Post, & Lesh, 1992), and whereby he or she acquires constancy of taste with respect to both location and measure. The second is about transformations and compensations between measures of the constituent elements of the mixture. The second conception builds on the first conception and establishes a foundation for the construction of the quantity of taste as an intensive quantity. An elaboration on the latter point follows.

Thompson (1994) and Kaput and Maxwell-West (1994) have distinguished among several levels of ratio conceptions. Thompson, in his investigation of the development of the concept of speed and its relation to the concept of rate, has made a distinction that it is not based on situations

\footnote{At least in our culture. Confrey (1988, 1989, 1994) suggested an alternative construct, the construct of splitting, which she argues can, and needs to, be developed as a complement to the additive world, but in the current mathematics curricula it receives inadequate nurturing.}
(a ratio is a comparison of like quantities [e.g., pounds vs. pounds], and a rate is a comparison of unlike quantities [e.g., distance vs. time]) but on the mental operations by which people constitute multiplicative situations:

The definitions of rate and ratio that I use do not emphasize situations. Instead, they emphasize the fact that ratios and rates are the products of mental operations, constructed and implemented by someone conceiving a situation. Thus, whether a quantity is a rate or a ratio depends on who is conceiving the situation and upon how he or she happens to be thinking of it (Thompson, 1993).

The first two levels of ratio, as identified by Thompson, are particularly relevant to the quality of constancy of quantity: (a) "ratio, where the multiplicative relationship is conceived as being between two specific, non-varying quantities;" and (b) "internalized-ratio, where the result of the relationship is fixed as well as the quantities being related, but the values of the related quantities vary" (Thompson, 1994). For example, as a ratio, a "per" statement such as "3 cups of orange concentrate per 4 cups of water" is conceived as a comparison of the two collections per se. As an internalized ratio, the same per statement would be conceived as representative of all ratios between two collections of the same quantities—cups of orange concentrate and cups of water—where the values of the collections vary multiplicatively (e.g., from "3 cups of orange concentrate per 4 cups of water" to "6 cups of orange concentrate per 8 cups of water").

A crucial difference between a ratio conception and an internalized-ratio conception is in the constancy of the result of the relationship: A person whose conception of the quantity of taste is at the ratio level would anticipate the taste of a mixture to vary when the amounts of its compared quantities (the constituent elements of the mixture) vary, without taking into account the kind of variation and whether the variation is multiplicatively preserved. However, if the person's conception is in the internalized ratio level, he or she would have a sense of constancy of the result of the relationship. Thompson's definitions allude to the notion of constancy of the result of the relationship but in the numeric sense; that is, it concerns the multiplicative relationship between the measurement values of the compared quantities. For example, a child who conceives the quantity of taste as an internalized ratio would understand that a mixture with 40 oz. of water and 24 oz. of orange concentrate will taste the same as any other mixture with $40 \times n$ oz. of water and $24 \times n$ oz. of orange concentrate where $n$ is a number conceived by the child as a multiplier. In different words, we may say that the numerical transformation $40 \rightarrow 40 \times n$ is compensated by the numerical transformation $24 \rightarrow 24 \times n$, which leaves the ordered pairs, or ratios, $(40, 24)$ and $(40 \times n, 24 \times n)$, equivalent. Constancy of taste, as we have defined earlier, may be acquired before the child has reached the level

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4As is well known, the child may not conceive of a decimal number or irrational number as a multiplier.
of thinking in terms of numerical transformations and numerical compensations. Her or his conceptual basis for constancy of taste may be that random samples of the same mixture taste the same because some sort of a trade-off between the amounts of constituent elements in the samples is preserved and that such a trade-off can occur due to the uniformity of the diffusion between the constituent elements. In this respect, we may talk about constancy of the result of the relationship in the intuitive sense, which, following Thompson, may be called intuitive internalized ratio.

The point of this discussion is that the acquisition of constancy of taste is hypothesized to emerge from a conflict between the child’s additive world and her or his physical experience with taste. This conflict leads to a construction of the notion of uniformity of diffusion of liquids. With this notion, the child can reason about the variation in the sample size in terms of a trade-off between the amounts of the constituent elements of the sample. At first, this trade-off is intuitive, in the sense that it is not represented in terms of numerical variations of the measurement values of the constituent elements; later, it grows into numerical transformations and compensations.

The ability to think in terms of numerical transformations is not necessarily a culminating point in the child’s conception of the quantity of taste. As research has widely shown, at the early stages children apply additive transformations to compare between tastes of different mixtures, and only in later stages are they able to reason in terms of multiplicative transformations. Nevertheless, thinking about quantities in terms of numerical transformations, additive or multiplicative, is an advanced stage in the child’s quantitative development. It is a stage where the quantity is conceived as a (numeric) relation (additive or multiplicative) between two quantities. We may talk about a culminating point in a child’s conception of a quantity only when the child conceives a quantity as multiplicative relation at the level of internalized ratio and, further, objectifies that relation into a conceptual entity (Greeno, 1983; Harel & Kaput, 1991). This happens when the multiplicative relationship becomes a numerical measure, in which case the quantity may be called a multiplicatively measurable quantity. How might such an ability develop with the quantity of taste and with other quantities? More specifically, how does a conception develop from an intuitive internalized ratio to a numeric relation to a multiplicatively measurable quantity? There is much to say about this process, but it is beyond the scope of this paper. It is an important question for research on the development of the additive and multiplicative conceptual fields and the links between them.

OBJECTIVE AND DESCRIPTION OF THE STUDY

Objective of the Study

This analysis was the motivation for the study reported in this paper, in that it led us to suspect that the acquisition of taste constancy may not be as
trivial as it appears, because it is conceptually equivalent to the acquisition of the quantity of taste as an internalized ratio. The purpose of this study was to investigate children’s anticipatory scheme of taste constancy; that is, children’s ability to compare the tastes of different samples of the same mixture without carrying out the actual action of tasting the samples. In an important respect, this study’s objective is not different from that of any of the studies that used mixture problems to investigate the concept of proportionality (e.g., Noelting, 1980a, 1980b). Their focus, too, was on children’s anticipatory scheme, but of a different kind, that which anticipates the relative strength of two samples of different mixtures based on the relative amounts of the ingredients that constitute the two mixtures. However, these studies, including those with very young children (e.g., 6 year-old children; see Tourniaire, 1986), seem to have taken children’s conception of taste constancy for granted.

It should be emphasized that this study concerns merely the psychological aspects of taste constancy conception, not its epistemological or instructional aspects. Namely, the study’s intent was to explore the state of children’s anticipatory scheme at the time they were interviewed, not the modification of this scheme. Although this latter aspect was alluded to in the theoretical analysis presented earlier, it was not part of the experimental component of the study. Because of this limited focus, our questions to the children centered on their anticipation of the relative strengths of the taste of samples taken from the same mixture. We did not intend to ask children questions that can potentially bring them into a cognitive conflict whereby they may modify their anticipatory scheme of taste constancy. For example, we did not have children do some testing and compare their experiential judgment to their anticipatory judgment; that type of investigation goes beyond the scope of this study. This is not to say, however, that the children who participated in this study did not alter their anticipatory scheme of taste constancy as a result of our interaction with them through the clinical interviews. Clinical interviews, as all know, are a learning experience for both the interviewee and the interviewer.

Subjects and Tasks

The instrument used in this study included a written test and one-on-one interviews. The entire class of 16 sixth-grade children in a private school in Indiana were given a written test that consisted of 11 problems. Only the second problem (see Figure 1) in this test dealt with constancy of taste; the rest dealt with the following topics: multiplication and division (à la Fischbein, Deri, Nello, & Marino, 1985), multiplicative invariance (à la Harel, et al., 1992), and the concept of unit (à la Behr, Harel, Post, & Lesh, 1992). None of these topics is directly relevant to constancy of taste. The test lasted about 60 minutes.

The written test was discussed in advance with the class teacher. She commented about the soundness of the problems and the material covered in her
class on each of the problem topics. She indicated that taste constancy was never addressed in her class and, as was confirmed by the class science teacher, was not addressed in the science class. Also, we learned from this teacher, the class science teacher, and the fifth-grade teacher about the experience the class had with lessons directed toward proportional reasoning: During the fifth grade and until the time this study was administered, the class dealt with only two instances of proportionality in physical phenomena: (a) balance-scale moment and (b) ratio comparisons of orange juice mixtures. In the latter, the class was presented with two sequences of numbers—one sequence representing numbers of cups of water, the other representing numbers of cups of orange concentrate—and told that all the ratios of corresponding numbers from the two sequences are equal (see Eicholz, O’Daffer, & Fleenor, 1981, p. 274). The general impression—gathered by the first author of this article in numerous class observations and interactions with teachers and children in the school where this study was conducted—was that the school puts a heavy emphasis on computations and procedural knowledge. Thus, despite the ubiquity of the concept of proportion throughout the elementary mathematics curriculum, children’s experience with this concept is primarily procedural rather than relational.

The interviews began a week later and were completed within two weeks. In these interviews, children were asked about their solutions to problems from the categories above. Eight of the sixteen children who took the test were interviewed on the taste constancy problem (see Figure 1). Each child was interviewed individually for 30 minutes, and four of these children were interviewed again for an additional 15 minutes that were devoted exclusively to the taste constancy problem. (Concerning the selection of the interviewees, we say more below.)

After a short introductory conversation with the child, the interviewer returned the child’s written test. No marks or comments were written on the test except those made by the child when he or she took the test. The child was told that he or she might look at the written responses at any time and that our aim was to understand the way he or she thought about solving the problem. During the interview, when a hesitation on the part of the child was noticed, the interviewer suggested that the child feel free to change the answer if he or she wished to do so. In each problem, the interviewer began by asking the child to describe the problem in her or his own words. For example:

\[ I: \] OK. So can you tell me what the problem was?
\[ AM: \] Uh, the 7-oz. glass tastes the same as the 4-oz. glass.
\[ I: \] Yeah, but let’s say you want to tell your friend about this problem, OK? Can you describe the problem to your friend?
\[ AM: \] Um, maybe.
\[ I: \] OK. Can you try? Let’s say your friend has never heard about this problem before—would you try to explain it to him?
\[ AM: \] Well there is 40 oz. of water and 24 oz. of orange concentrate and a 7-oz. glass of California orange juice and the 4-oz. glass of California orange juice. And, um, which one would taste more orangy?
The task posed to the children is in Figure 1. It presents a drawing of a carton of orange juice, called California Orange Juice, with a label describing the amounts of its ingredients: “40 oz. of water and 24 oz. of orange concentrate.” Below this, there is a drawing of two glasses with corresponding labels: “7-oz. glass of California Orange Juice” and “4-oz. glass of California Orange Juice.” The regions of the glasses are shaded in, indicating that the glasses are filled up to the top. In the end, the child is asked: (a) Would the orange juice from the 7-oz. glass taste the same as the orange juice from the 4-oz. glass? (b) If they wouldn’t taste the same, can you tell which one would taste more orangy? (c) Explain.

Figure 1. The written test.

RESULTS AND DISCUSSION

Children’s Final Answers

On the written test, eight of the children (half of the subjects) answered that the two glasses would not taste the same (“Not the same”): four children
answered that the 4-oz. glass would taste more orangy ("4-oz. glass"), three children answered that the 7-oz. glass would taste more orangy ("7-oz. glass"), and one did not specify which of the two glasses would taste more orangy ("No specification"). The distribution of answers of the other half of the subjects (8 children) was: one child answered "they would probably taste the same" ("Probably the same"), one child didn’t answer ("No answer"), and the other six answered that the two glasses would taste the same ("The same"). A summary of the distribution of these responses appears in Table 1.

<table>
<thead>
<tr>
<th>Type of response</th>
<th>Number of responses</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not the same</td>
<td>8</td>
</tr>
<tr>
<td>4-oz. glass</td>
<td>4</td>
</tr>
<tr>
<td>7-oz. glass</td>
<td>3</td>
</tr>
<tr>
<td>No specification</td>
<td>1</td>
</tr>
<tr>
<td>Probably the same</td>
<td>1</td>
</tr>
<tr>
<td>No answer</td>
<td>1</td>
</tr>
<tr>
<td>The same</td>
<td>6</td>
</tr>
</tbody>
</table>

Eight children were interviewed. The selection of these children was intended to cover the variety of responses given by the children in the written test. Thus, five representatives were selected from the eight children who answered "Not the same" on the written test, one representative from the six children who answered "The same," the one who answered "Probably the same," and the one who did not answer the question. When interviewed, the last three answered "The same," and among the other five only one child changed his answer to "The same"; the remaining four either stayed with the same answer or went back and forth between the "7-oz. glass" answer and the "4-oz. glass" answer. For example, on the written test, Student AN answered "4-oz. glass," but on the interview, in response to the same question asked on the written test, she was unsure which glass would taste more orangy. At first she thought "4-oz. glass"; then she hesitated and changed to "7-oz. glass."

The distribution of the written answers—four answers "4-oz. glass" and three answers "7-oz. glass"—does not support the possibility that children misunderstood the question by interpreting the "more orangy" in the problem question with "a larger amount of orange juice." Such an interpretation would lead to the "7-oz. glass" answer but not to the "4-oz. glass" answer. Further, during the interview the question "Do they taste the same or is one more orangy" was rephrased, making it explicit by asking about the tastes of random samples and about the strength of the mixture's flavor. For example, the interviewer asked:

Let's say we have two glasses here, a 7-oz. glass and 4-oz glass. Both are full of orange juice from this carton [pointing to the carton in Figure 1], OK? And I
take a teaspoon from the 4-oz. glass, drink it, and then take the same teaspoon from the 7-oz. glass and drink it. My question is which one would have a stronger flavor, or would they probably have the same flavor?

With one exception, this question did not cause the children to change their written answers from “Not the same” to “The same.” The one child who did change his answer did so before the interviewer rephrased the question.

Children’s Explanations

The child who on the written test answered “The same” kept with the same answer during the interview. For her, the issue was self-evident and needed no explanation:

*EY:* Well, they both taste the same, cause, like, well it’s just that it doesn’t really matter how much is in it if the ingredients are mixed, cause it doesn’t make a difference at all.

The child who answered “Probably the same,” changed to “The same,” though not with complete confidence at first.

*I:* Can you tell me what the problem is? Let’s say I have never seen this problem before and you want to tell me about it.

*RN:* OK. Well, you wanted to know, you saw that there’s a 7-oz. glass of California orange juice and then there’s a 4-oz. glass of it. And you want to know if you’ll have the same, if it will taste the same.... I think they would probably taste the same, because um ... [long pause].... I’m mostly sure but there’s an inch of no faith there ... Um hum. Well, the way I see it, they would have the same because this is in a whole entire, you know, the whole entire carton, where there are 40 oz. of water and there are 24 oz. of orange concentrate ... and so, it really [does] not depend on how much you pour because this will be like, it won’t always be 40 oz. of water if you subtract how much you take out ... and it doesn’t do anything to the taste because this number will keep subtracting depending on how much you put inside the glass.

The last part of this response is particularly interesting because it hints at RN’s thinking of the trade-off between the amount of water and the amount of orange concentrate that is preserved when the orange juice is poured into the glasses. Using our theoretical analysis, we may say that RN is at the stage of understanding the quantity of taste as an *intuitive* internalized ratio.

The written explanations, despite their brevity, provide some clues about the rationales children used for their judgments that the two glasses wouldn’t taste the same. For example, *ZK* explains that the 4-oz. glass would taste more orangy because “it has less water in it but more orange per ounces.” *AM* explains that the two glasses would taste different because “the 7-oz. glass has room for more water and less orange stuff, but the 4-oz. glass has less water and more orange stuff.” *JH*, on the other hand, answered that the 7-oz. glass would taste more orangy “because the glass is bigger, so it would hold more orange.” It seems that these children believe that the taste of a mixture depends on the volume occupied by the mixture. For some children, a smaller volume would allow less water to get in,
which would leave more room for orange concentrate, and as a result, the mixture would taste more orangy. For other children, the opposite is true: a larger volume would allow more water to get in, which would leave less room for orange concentrate, and therefore the mixture would become more watery. The analysis of the interviews revealed the following three factors in the children's reasoning about constancy of taste, the first of which is consistent with the written responses just described:

1. The relative volumes of the mixtures to be tasted
2. The uniformity of the liquid to be tasted, that is, whether it is thought of as consisting of a single ingredient or more than one ingredient
3. The numerical data of the problem

These factors will be discussed in the same order in the next three subsections.

Relative volumes. AM's written response was that the two glasses "would taste different" without saying explicitly which glass would taste more orangy. As was indicated earlier, his explanation to this was that "the 7-oz. glass has room for more water and less orange stuff, but the 4-oz. glass has less water and more orange stuff." From this we inferred that he thought the 4-oz. glass would taste more orangy. On the interview, however, his first response was "7-oz. glass." To defend his answer, he used an argumentation similar to that he had used on the written test, but in the opposite direction:

I: OK. And what was your answer?
AM: I said they'd taste different.
I: Why is that?
AM: Um, because a 7-oz. glass has room for more of both, and the 4-oz. glass has not enough room for orange, they have more orange in there, 'cause it has more room than that one.
I: So which one would taste more orangy?
AM: The 7-oz glass, because it has more room for orange stuff, the orange concentrate.

When the interviewer attempted to point to a flaw in his reasoning, he received no response from AM:

I: But it also has more room for water.
AM: Yeah. [No further response.]

AM's judgment of the relative strength of the taste of the two glasses is based on the relative size of the glasses. But it must be noted that his judgments and explanations are inconsistent: whereas in the written test he predicted that the relative strength of the two glasses' taste is in an inverse relation to the relative size of the two glasses, in the interview he arrived at the opposite conclusion. What's particularly important about AM's responses is that his explanations do not involve any expression of the uniform diffusion between liquids—an indication, according to our theoretical perspective, that AM has not constructed the notion of uniformity of diffusion between liquids in a mixture. As a result, AM is not able to reason about the variation in the sample size in terms of a trade-off between the
amounts of the constituent elements of the sample. Such a reasoning, as we have explained, constitutes the conception of the quantity of taste as an intuitive internalized ratio—an equivalent conception to the conception of constancy of taste.

Would AM’s response be different had he reenacted the situation with a personal involvement? This is what the interviewer intended to find out when he asked:

OK. I want you to imagine for a second a situation where you have this container [pointing to the carton of orange juice in Figure 1] and you were with your friend; you had just finished playing a basketball game. You both were thirsty. You got two glasses, a small one and a big one. You gave your friend the big one, and you got the small one, and you fill up the two glasses with California Orange Juice. Who do you think is going to have his orange juice more orangy, or are they going to taste just the same?

This approach of personalizing the situation was used with the other interviewees as well. In all cases, including AM, it did not cause children to change their answers.

AM was interviewed again 10 days later. This time the interviewer specified more explicitly that the concern is the taste of the samples taken from the container. AM was told in this second interview that the amount of mixture taken from one glass and put in the mouth is equal to the amount of mixture taken from the second glass and put in the mouth. This was done by rephrasing the question asked in the written test as follows:

I: Let’s say we have two glasses, a 7-oz. glass and a 4-oz. glass. And you fill them up with the California orange juice. If I took a teaspoon from the 4-oz. glass and the same teaspoon from the 7-oz. glass, which teaspoon do you think is going to taste more orangy, or are they probably going to taste the same?

As can be seen, this did not result in a change in AM’s reasoning about the situation.

AM: Um, the 4-oz. glass.

I: From the 4-oz. glass?

AM: Yeah.

I: OK. Could you please explain this to me?

AM: No [pause]. The 7-oz. glass, it will have more water in it ... and more orange stuff ..., and it [pointing to the 4-oz. glass] will have less water and less orange.

We did not ask AM to predict the relative taste of two glasses of the same size. But, when we asked CP this question we received a response that, if analyzed in terms of our logic, would be found totally inconsistent:

CP: [In response to the question about the different sized glasses] They wouldn’t taste the same.

I: Which one will have a stronger flavor?

CP: The 4-oz., because there is, there would be more, because it’s wider⁵ than that.

⁵From a further conversation with CP it turned out that he saw the 4-oz. glass to be wider than the 7-oz. glass.
I: More what?
CP: More of the orange and water.
I: Uh huh. Let's say that I took a teaspoon of orange juice from the 7-oz. glass ...
CP: Yeah ...
I: ... and gave it to you to drink. Then I did the same thing from the 4-oz. glass. I took a teaspoon of orange juice from the 4-oz. glass and gave it to you to drink.
CP: Yeah.
I: Which teaspoon would have a stronger flavor, or probably they would have the same flavor. What do you think?
CP: That would have stronger flavor I think.
I: Which one?
CP: The 4-oz. of California.
I: The one from the 4-oz. glass?
CP: Yeah.
I: How would you explain your answer?
CP: OK. Um, it say[s] 40 oz. of water and 24 oz. of orange concentrate.... Or something. And when you pour this glass up, I forget, I forget if I, if I found out how much both of these would be in it, but ...
I: You can do that now if you like.
CP: OK. I forget how I did this, but um, it's like for the 7-oz. glass of California orange juice, um, I forget what it is but I think it has something to do with the ingredients.
I: OK. Now, what if you had the same size of glasses? Let's say both were 4-oz. glasses. Which one is going to have a stronger flavor?
CP: Well, I'd think they'd both taste the same.

CP's explanation seems to say that the two glasses do not taste the same because they are of different size; and, conversely, had the two glasses been of the same size they would have the same taste. At the same time, CP thinks that the two teaspoons, one taken from the 4-oz. glass and one from the 7-oz. glass, do not taste the same. If CP thought that the two teaspoons are of the same size, this is clearly a contradictory response (from our viewpoint, not the child's, of course).

We must emphasize that logical inconsistencies are not the issue here. We highlighted this event to stress that children's conception of taste constancy is complex and our knowledge of it is meager. In addition to inconsistency, the contradictory features of uncertainty and self-evidence were also characteristics of children's responses. During the interviews children would repeat their arguments without being able to elaborate much further. The tone of their responses leaves the impression that they view their conclusions as self-evident, needing no further explanation. Although they were certain that the two glasses wouldn't taste the same, they were uncertain as to which

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6Although the interviewer did not say explicitly that the two teaspoons were of the same size, he did say that the same thing done with the 7-oz. glass is done with the 4-oz. glass, which implies that the same teaspoon was used in both cases.
glass would taste more orangy. In some cases the same child would use both arguments, one to claim that the 4-oz. glass would taste more orangy and one that the 7-oz. glass would taste more orangy.

Uniformity. Going back to AM, it turned out that the fact that the California orange juice was made of different ingredients was crucial to his reasoning about the problem:

I: Let’s say it wasn’t orange juice, let’s say it was Coke.
AM: Yeah.
I: You fill up the 4-oz. glass and the 7-oz. glass with Coke. And you took a teaspoon from the 4-oz. glass, and a teaspoon from the 7-oz. glass. Are the teaspoons going to taste the same, or is one of them going to have a stronger flavor of Coke?
AM: It will taste the same.
I: Why for Coke did you answer the two glasses would taste the same, but for orange juice you answered they wouldn’t taste the same?
AM: Well, it doesn’t matter, like if it’s, it’s just Coke. It doesn’t have water and orange stuff. It just comes from the same thing. (emphasis added)

CP, on the other hand, didn’t think that Coke is “ingredients free.” As a result, he insisted that Coke too would be subject to the same rule as orange juice.

I: Let’s say that it wasn’t orange juice, it was Coke.
CP: Um hum.
I: OK. And you filled up the 4-oz. glass with Coke, and the 7-oz. glass with Coke.
CP: Um hum.
I: Which one is going to have a stronger flavor of Coke?
CP: I think it’d be the 4-oz. glass.
I: Why is that?
CP: Because this is wider and then um, so then, it would be like more, because longer doesn’t mean that it’s bigger than the short one that’s wider.

The last response was incomprehensible to the interviewer, who then rephrased the question by removing the numerical data and personalizing the situation for CP:

I: Now, um, imagine that you open your refrigerator, and you take out a bottle of Coke. You have two glasses, one for your friend and one for you, a small one and a big one. Forget sizes. We just have a small glass and a big glass.
CP: Yeah.
I: And you filled them up to the top. OK?
CP: Yeah.
I: And you just take a sip from one, OK? Then your friend takes a sip from the other one.
CP: Yeah.
I: Which one is going to have a stronger taste of Coke?
CP: Well, I think it would be the 7-oz. thing. Because it’s got more in it.
I: But you are not going to drink the whole thing. You are just taking a sip. (emphasis added)
CP: Yeah. There would be more ingredients that would like be circulating in it [the 7 oz. glass].

Several things we have noticed from this excerpt. First, CP stayed with his answer despite the removal of the numerical data and the personalization of the situation. Second, even when the interviewer left no doubts that the question is about samples ("You are just taking a sip"), CP continued to insist on his conclusion that the two glasses wouldn't taste the same. Third, his thinking about the diffusion between the ingredients is quite explicit (see the last sentence in the last excerpt). Fourth, and last, CP’s answers are inconsistent: first he answered “4-oz. glass” and a few minutes later changed to “7-oz. glass.” An attempt by the interviewer to understand the source of this inconsistency led nowhere, except for the fact the CP saw the 4-oz. glass to be wider than the 7-oz. glass, and that apparently was important to him, which may have to do with the “relative volume” factor discussed earlier:

I: But I thought that you said...but before you said that the 4-oz. glass.

CP: Um, uh, well, I, it looks like that one [the 4-oz. glass] is wider so there would be more ...

I: Wider? Where do you see it’s wider? Can you point where it’s wider?

CP: It looks wider right there [pointing to the bottom of the glass].

I: From here to here [from the left edge of the glass to its right edge]?

CP: Yeah.

I: OK. It’s wider than what?

CP: From this way, it [the 4-oz. glass] looks wider than that [the 7-oz. glass].

I: Well, this is really the same width, except that this one [the 7-oz. glass] is higher.

CP: Yeah, OK, I see. OK. Well then I think it be the 7-oz....

For “liquids without ingredients,” CP thinks that different samples would have the same taste:

I: Um hum. OK. What about water? What about if instead of Coke and orange juice we have water?

CP: It would be the same.

I: Why is that?

CP: Because water doesn’t have like really all these ingredients in it, it’s just water (emphasis added).

We asked the Coke question in order to get better insight about the uniformity factor in children’s conception of taste constancy. We conjectured that with Coke, because of its strong homogeneous appearance, children would answer “The same.” We were interested in how children would compare and contrast the question with Coke to that with orange juice. As we have seen, AM thought that the same rules that applied to Coke cannot apply to orange juice because the first, in contrast to the latter, is “ingredients free.” CP, however, believed that both Coke and orange juice consist of more than one ingredient and therefore are subject to the same “rules.”
With one child, AN, her comparison between the Coke question and the orange juice question evoked a cognitive conflict, though not spontaneously:

**I:** What was your answer for the orange juice problem?

**AN:** That the 4-oz. glass would taste more orangy.

**I:** And what was your explanation?

**AN:** Well, I can’t really remember what my explanation was.... I think I divided 4-oz. into 40 oz. of water and then 4 into 24 oz. to get 10 and 6.... And then 7 into 40 to get 5 and 5 sevenths, and 7 into 24 to get 3 and 3 sevenths.

**I:** Let’s say I have these two glasses here filled up with the orange juice, OK, and I take a teaspoon from the 4-oz. glass and drink it. Then I take the same teaspoon from the 7-oz. glass and drink it. Which teaspoon is going to have a stronger flavor or are they probably going to taste the same?

**AN:** Well, uh, now I don’t really think that the 4-oz. glass would but I’m not real sure that the 7-oz. glass would either, ’cause I’m not sure how I figured it out.... I guess I’d still think the 4-oz glass.

**I:** Let’s say it was not orange juice. Let’s say it was Coke. You fill up the 4-oz. glass and the 7-oz. glass with Coke and you take a teaspoon from the 4-oz. glass, and a teaspoon from the 7-oz. glass. Are the two teaspoons going to have the same flavor or is one of them going to be of a stronger flavor?

**AN:** Well, ah, ah, I guess they’d probably have the same flavor (emphasis added).

**I:** Why is your answer for Coke different from your answer for orange juice?

**AN:** I guess they’d probably have the same for the orange juice too. It’s just you’re pouring 3 more oz. in to one glass and so it’s, it’s, well I don’t know, I guess it would just kind of taste the same because it’s got the same amount of stuff in it pretty much and then if you just taste it and you tasted it to see which one tastes stronger I think they’d probably taste about the same (emphases added).

**I:** So what made you change your answer?

**AN:** Ah, well I guess I kind of tried the way to do it about the Coke, and the example about the Coke and I guess that from when I’ve done that before that they tasted the same pretty much, so ... if my answer for Coke was different than the answer for orange juice there is something wrong, ’cause they both have the same amount of ounces and stuff so they probably both ought to have the same answer (emphases added).

It is clear that the Coke question shook AN’s confidence about her previous answer. In the next subsection we will see another source for AN’s uncertainty of her answer.

**Numerical data.** The following dialogue continues the one just presented. It shows that the numerical data that appear in the problem statement were a serious source for AN’s uncertainty.

**I:** So what do you think happened before that your answer was that the two glasses wouldn’t taste the same.

**AN:** Well, because when I divided the, like, 4 into 40 and 24, and 7 into 40 and 24, then I got more ounces than could actually fit into the glasses. [She was looking at her written response, where she wrote by the 4-oz. glass figure, 40 + 4 = 10 and 24 + 4 = 6; and by the 7-oz. glass figure, 40 ÷ 7 = 5½ and 24 ÷ 7 = 3½.] For the 4-oz. glass you can’t fit 16 [10 + 6 = 16] oz. into a 4-oz. glass, and like for the 7-oz. glass you can’t fit 9 oz. and a 7th of an ounce [5½ + 3½] ...
Notice that AN rejected her first conclusion that the two glasses wouldn’t taste the same, not on the grounds of taste constancy but on the grounds of a conflict she encountered between the results of her computations—which she interpreted as representing the amounts of each ingredient in each glass (i.e., 40 \div 7 = 5\text{/7} and 24 \div 7 = 3\text{/7})—and sizes of the glasses (i.e., the amounts of ingredients in her calculations exceeded the size of the glass). In the beginning of the next excerpt the interviewer tried to suggest to AN a hypothetical situation where such a conflict wouldn’t occur. It is not clear whether AN understood the interviewer’s suggestion, and if she did how it influenced her reasoning. It is clear, however, that she (hesitantly) changed her conclusion.

I: What if you got small numbers? I mean what if when you divided you got small numbers so it would fit? Let’s say here, for the 4-oz. glass, you got 3 and 1 rather than 10 and 6, and in the 7-oz. glass you got 6\text{/7} and 1\text{/2} rather than 5\text{/7} and 3\text{/7}, so they could fit.

AN: I think they’d probably taste the same.... Not positive on it.

I: What makes you hesitate about it?

AN: It’s pretty much because I’m not sure how I’m supposed to do the problem so I’m not sure if I’m doing it right or if I’m doing it wrong. So I can’t be sure as to whether or not I’m answering the question right or wrong.

I: OK. Let’s say you don’t know at all that you have 40 oz. water and 24 oz. orange concentrate. No one told you the amounts of the ingredients. But you know that orange juice is made of water and orange concentrate. And you took two glasses, a small one and a big one. No numbers are involved here, OK?

AN: Um hm.

I: Now, you fill these two glasses up, and the question is if you took a teaspoon from one glass and a teaspoon from the other glass, which teaspoon would have a stronger flavor or are they going to have the same flavor?

AN: I guess they’d have the same flavor, just cause if I don’t know what the ingredients are or how much is in each one of them I’d think that they’d probably both have the same flavor no matter what the size of the glass was (emphasis added).

I: Uh huh. But if you do have the numbers here, OK, the amount of water and the amount of orange concentrate, and you know that one size is 4 and one size is 7, would your answer be different?

AN: I don’t know. I’m kinda thinking that maybe the 7-oz. glass might have a stronger taste kind of.... Just because it’s got a bigger number to it, so you think you could fit more stuff into it (emphasis added).

What’s amazing about this response is that for AN, whether the glasses would taste the same or not depended on whether she knew the actual sizes of the glasses and the amounts of the ingredients. This may have to do with AN’s belief about mathematics that when numbers are involved in a problem, computations must be carried out to solve the problem. Her computations led her to conclude that the two glasses wouldn’t taste the same (despite the contradiction she encountered with these computations; see above), but she couldn’t ignore this fact despite her feeling that the two glasses should taste the same. Knowing the current state of school
mathematics and its assessment practices (see Lesh & Lamon, 1992), this kind of belief should be expected. But the robustness of the belief is surprising. What this robustness may indicate is that her conception of taste constancy has not yet reached a firm state of intrinsic stability. Metaphorically speaking, in the competition between her beliefs about how mathematical problems should be solved and her conception of taste constancy, the former wins.

CONCLUSIONS

According to the theoretical perspective presented earlier, the conception of constancy of taste is linked in a fundamental way to the conception of internalized ratio, which constitutes the conception of the quantity of taste as a ratio that is invariant under multiplicative transformations of the ratio’s components. It is this conception that enables a person to anticipate that the taste of a mixture will not vary when the volume of the mixture varies. We have suggested that this anticipatory scheme emerges from a disequilibrium between the child’s additive world and her or his experiences with the quantity of taste. This scheme combines two interrelated conceptions: the conception of the uniform diffusion between ingredients of a mixture and the conception of multiplicative transformations and compensations between measures of the ingredients of a mixture.

The results just discussed make a claim about the existence of a phenomenon: constancy of taste is self-evident to us, but children’s acquisition of it cannot be taken for granted. Half of the 16 children who took the written test answered “Not the same;” 5 of them were interviewed, and all but one continued to hold the same conclusion. The distribution of the written responses together with the children’s responses to the rephrased question about the random samples (teaspoons and sips) in the interviews do not support the possibility that children may have misunderstood the question by interpreting the “more orangy” in the problem question as “a larger amount of orange juice.” The children continued to hold their conclusion even when the interviewer tried to personalize the situation for them by asking them to imagine themselves acting out the situation.

Children’s written explanations suggest that they believed the relative tastes of the two glasses depend on the relative volumes of the glasses (the “relative volume” factor). The interviews also pointed to this conception, and further revealed two boundary conditions associated with it: the “uniformity” factor and the “numerical data” factor. The “uniformity” factor refers to children’s belief that the taste of a liquid is invariant under a change of volume only if the liquid is viewed as “ingredients free.” Further, if the liquid is conceived by the children as consisting of more than one ingredient, they believe that the relative strength of the taste of different samples of the same liquid is dependent on the relative volume of the containers from which the samples are taken. In the eyes of an adult, the
Uniform Diffusion Principle explains the irrelevancy of the “uniformity” and “relative volume” variables to constancy of taste. This may suggest that the children who included these variables in their explanation of relative taste of samples have not yet constructed the Uniform Diffusion Principle. The “numerical data” factor refers to children’s belief that taste constancy holds only when the context is “number free.” The latter conception, we have suggested, indicates the yet-intrinsically-unstable conception of taste constancy.

These results were obtained with sixth-grade children. Younger children are likely to demonstrate an even more primitive conception of constancy of taste. Research studies on the proportion concept have used mixture problems with first and third graders. Moreover, many of these studies provided problem data in numerical terms. On the one hand, a central purpose of these studies was to investigate children’s conception of proportion at the intuitive level, that is, children’s conception of intuitive internalized ratio. On the other hand, these studies seem to take children’s conception of taste constancy for granted. If, indeed, constancy of taste and intuitive internalized ratio are conceptually equivalent, as our analysis suggests, the results of this study raise questions about the validity of the findings of these other studies.

Limitations

As we have indicated, this study’s only objective was to draw attention to a phenomenon, not to investigate its epistemological factors and patterns. For this, further investigations with different approaches and methodologies are needed. Our study has some noticeable limitations. For example, our methodology is limited in that it does not control for different variables (e.g., inclusion of numerical data in the problem, inclusion of physical instruments), and the interaction with the children was relatively short. It is reasonable to note these limitations as well as concerns about wording that should have been used or questions that should have been asked. To put this in context, we acknowledge our hesitations and uncertainty during the preparation of the tasks and interviews. By this we do not intend to justify the limitations, only to stress that this study’s sole aim was to take a first look at a phenomenon that was difficult for us to comprehend. As is explained below, the difficulty lay in the fact that our prediction of the phenomenon was solely theoretical. From our own intuitive perspective based on common experience this phenomenon made little sense.

On the one hand, looking at the phenomenon from a theoretical point of view there was reason to believe, as was discussed in the Theoretical Analysis section, that constancy of taste must involve the construction of the quantity of taste as an internalized ratio, in conjunction with the construction of the Uniform Diffusion Principle. Both of these conceptions are far from being self-evident. Thompson (in press) and Kaput and Maxwell-West (in
press) have documented the difficulty children have with the notion of internalized ratio. Surprisingly, research in science education has not, to our knowledge, addressed the conception of uniform diffusion among liquids. The acquisition of this conception is, undoubtedly, part of a more global process of understanding “uniformity” in space. Uniformity, as can be seen from the following demonstration by Zeldovich (1992) in his article on the universe for high school students, is a difficult notion:

Air in a vessel is uniform. There are the same number of oxygen molecules and the same number of nitrogen molecules in each cubic centimeter. But if we take tiny volumes—for example, cubes with edge $3 \times 10^{-7} \text{ cm}$, whose volume is approximately $10^{-20} \text{ cm}^3$—then they contain only one molecule of oxygen or nitrogen on average. So at any given moment there isn’t a single molecule in this cube or several cubes, while other cubes can contain one molecule of N, or one of $O_2$ or, perhaps, two or three molecules (of the same or different elements). We can see uniformity only when we consider rather large volumes. In the universe uniformity occurs in volumes (cubes) with edges greater than 300 megaparsecs [1 megaparsec $= 2 \times 10^{19}$ miles]. (p. 8)

On the other hand, from an experiential perspective, the idea of constancy of taste seemed to us clear and self-evident, and thus it seemed unnecessary to question children’s understanding of it. The question, “Do different samples of the same mixture taste the same?” sounded so trivial that it looked totally unreasonable that a sixth-grade child would answer incorrectly. But recalling Piaget’s (1960) enlightening remark that “in psycho-genetic development, a mental operation is deceptively simple when it has reached its final equilibrium, but its genesis is very much more complex” (p. 3), we were encouraged to pursue the experiment despite the uncertainty.

**Final Comment**

The conception of constancy of taste is not all or nothing. What this study shows is that in the context in which the question was posed, children did not demonstrate an understanding of constancy of taste. In different contexts—for example, if the question didn’t include numerical data—children’s responses might have been different. This by no means refutes the existence of the phenomenon this study has addressed.

We have considered the argument that the children responded the way they did because they did not understand the idea of “ideal conditions” (see the definition of taste constancy in the beginning of this paper). We reject this argument on the following ground: The notion of “ideal conditions” is an integral part of the conception of constancy of quantity, be it the quantity of taste, speed, price, or any other similar quantity. That is, children do not first construct constancy of quantity and later think of the “ideal conditions” under which it holds; nor do they do the reverse. Rather, the conception of constancy of taste (or any other quantity) includes the contexts and conditions where it is perceived to hold and those where it is perceived not to hold. These contexts and conditions are an integral part of the conception
Invariance of Ratio

Invariance of Ratio

itself. All the notions involved in constancy of taste—constancy with respect to location, constancy with respect to measure, “ideal conditions” where these constancies hold, and the contexts where they do not hold—are interrelated components of the same scheme. They do not develop chronologically, but concurrently, and are mutually dependent on each other.

To an adult who has already acquired constancy of taste, the whole matter may seem rather self-evident. To this we can only say that until we began this research it was not self-evident to us that constancy of quantity is not self-evident for everyone.

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