Graph Theory Introduction

Math Circle, Fall 2011

Graph theory has been instrumental for analyzing and solving problems in areas as diverse as computer network design, urban planning, and molecular biology. Graph theory has been used to find the best way to route and schedule airplanes and invent a secret code that no one can crack.

In the branch of mathematics called **Graph Theory,** a **graph** bears no relation to the graphs that chart data, such as the progress of the stock market or the growing population of the planet. Graph paper is not particularly useful for drawing the graphs of Graph Theory.

In Graph Theory, a **graph** is a collection of dots that may or may not be connected to each other by lines. It doesn't matter how big the dots are, how long the lines are, or whether the lines are straight, curved, or squiggly. The "dots" don't even have to be round! All that matters is which dots are connected by which lines.

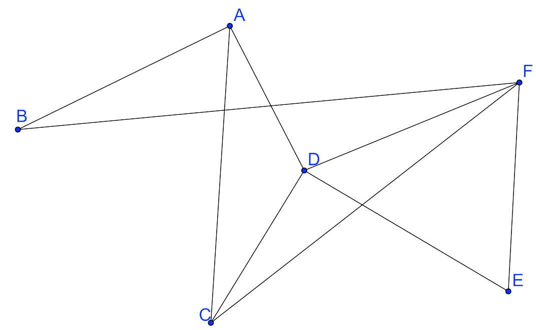
Graphs:

In a **simple graph**, two dots can only be connected by one line. If two dots are connected by a line, it's not "legal" to draw another line connecting them, even if that line stretches far away from the first one. Graphs where two dots are connected by more than one line are called **multigraphs**. A line that connects a point to itself is called a **loop**. Loops are also not allowed in simple graphs, but may or may not be allowed in multigraphs.

Simple graphs:

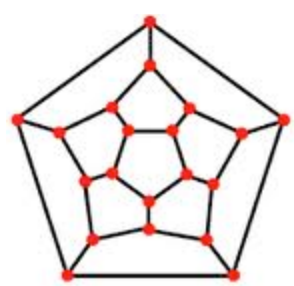
Multigraphs:

A graph is made up of dots connected by lines. A "dot" is called a **vertex**. When there is more than one vertex, they are called **vertices**. A "line" is called an **edge.** The **degree** of a vertex in a graph is the number of edges that touch it.

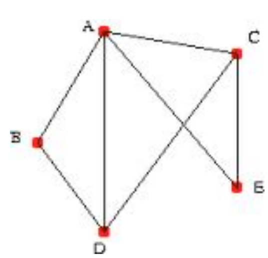


The **size** of a graph is the number of vertices that it has. We normally say "a graph with *n* vertices." What is the size of the previous graph?

A graph is **regular** if every vertex has the same degree. We may call the graph *k*-regular if every vertex has degree *k*.



A **path** is a route that you travel along edges and through vertices in a graph. All of the vertices and edges in a path are connected to one another. A **cycle** is a path which begins and ends on the same vertex. A **cycle** is sometimes called a **circuit**. The number of edges in a path or a cycle is called the **length** of the path. Is the length of the path also the number of vertices in the path?



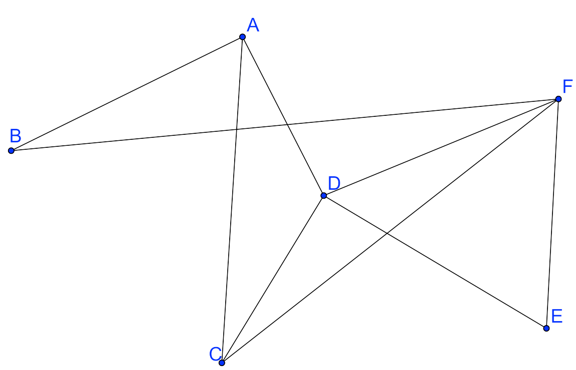
A graph is said to be **connected** if there is a path from any vertex in the graph to any other vertex. If a graph is not connected, we call each connected part a **component**.

Connected: Not connected:

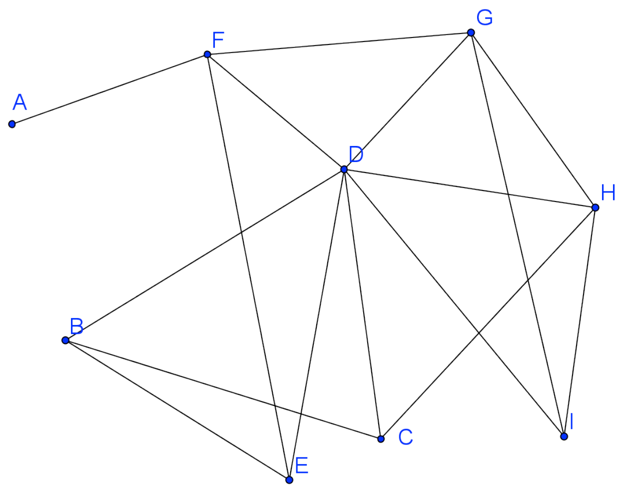
A **tree** is a connected graph with no cycles. How many edges does a tree have?

A **planar** graph is a graph that can be drawn so that the edges only touch each other where they meet at vertices. You can usually re-draw a **planar** graph so that some of the edges cross. Even so, it is still a planar graph.

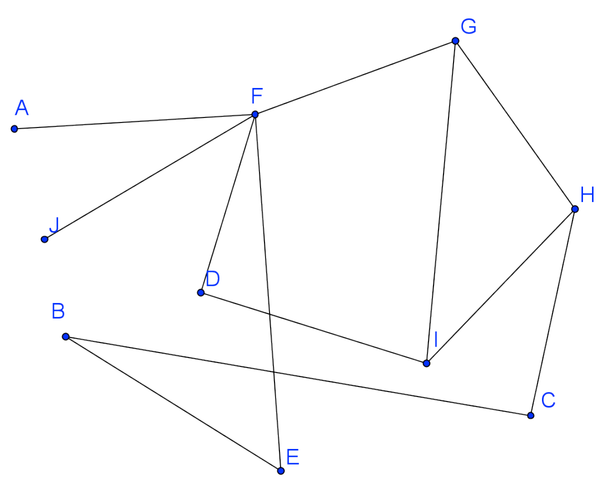
When it is drawn so that the edges cross, the drawing is called a **non-planar representation** of a planar graph. Is the following graph planar?



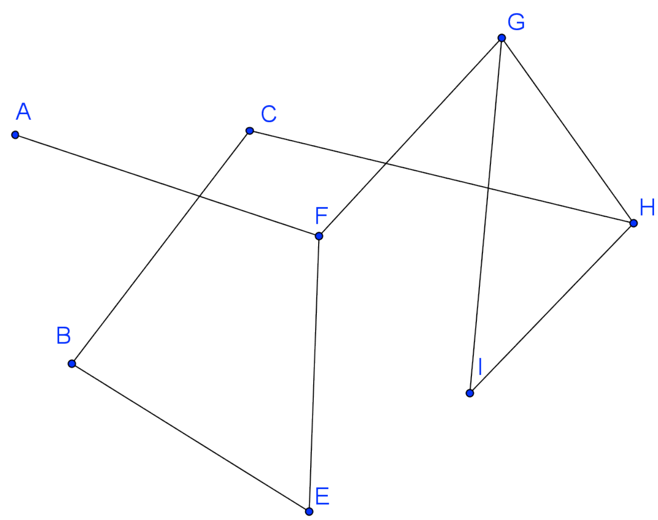
Some graphs are **nonplanar**. No matter how you stretch the edges around, you cannot redraw the graph so that none of the edges cross each other between the vertices. Is the following graph non-planar? How can we tell if graphs are nonplanar?



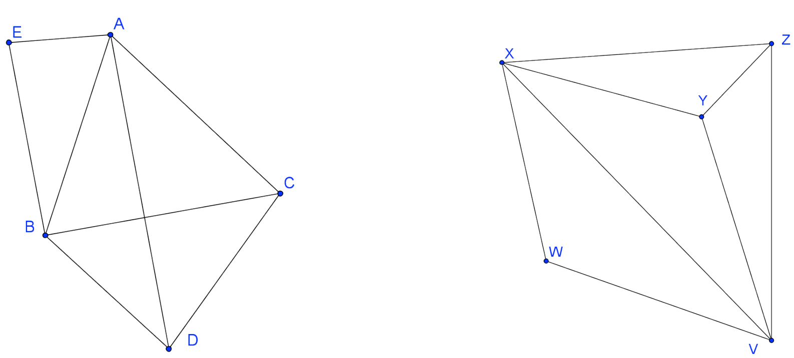
Distance in a graph isn't measured in inches or kilometers. This isn't surprising, because you don't do any measuring in inches or kilometers when you draw a graph in the first place. Still, when you look at a graph, you can see how it might be possible to say that some vertices are closer together then others. The distance between two vertices is a count of the number of edges along which you must travel to get from one vertex to the other vertex. If there is more than one path between two vertices, the number of edges in the shortest path is the distance. The number of edges in a path is called the **length** of the path.



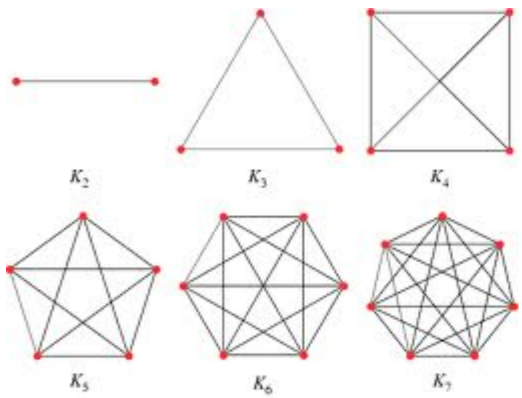
The **diameter** of a graph is the longest distance you can find between two vertices. When you are measuring distances to determine a graph's diameter, recall that if 2 vertices have many paths of different distances connecting them, you can only count the shortest one. An interesting problem in graph theory is to draw graphs in which both the degrees of the vertices and the diameter of the graph are small. Drawing the largest graphs possible that meet these criteria is an open problem. What is the diameter of this graph?



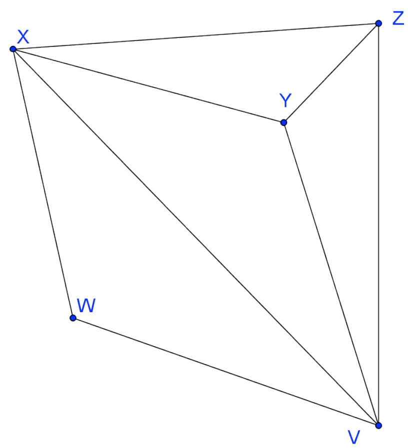
Two graphs are **isomorphic** if you can re-draw one of them so that it looks exactly like the other. To re-draw a graph, it helps to imagine the edges as infinitely stretchable rubber bands. You can move the vertices around and stretch the edges any way you like -- as long as they don't become disconnected. Sometimes it is very hard to tell whether two graphs are isomorphic or not. In fact, no one knows a simple method for taking two graphs and determining quickly whether or not they are isomorphic. Are these graphs isomorphic?



In a **complete** graph, every pair of vertices is connected by an edge. We denote it K_n, where n is the number of vertices. It is impossible to add an edge to a complete graph because every possible edge has been drawn. Complete graphs always have diameter **1**. Why?



In a graph, the **neighbors** of a vertex are all the vertices which are connected to that vertex by a single edge. The distance between two vertices which are neighbors is always 1.   What are the neighbors of vertex Y?



Problems

1. Below are some possible degree sequences for a graph with 8 vertices. For each sequence, draw a graph so that the given list describes the degrees of the 8 vertices in the graph or explain why you can't draw such a graph.

   a. 5, 5, 4, 3, 2, 2, 2, 1

   b. 4, 4, 3, 2, 2, 2, 1, 1

   c. 5, 5, 4, 4, 2, 2, 1, 1

   d. 5, 5, 5, 3, 2, 2, 1, 1

   e. 5, 5, 5, 4, 2, 1, 1, 1

   Can you generalize? Can you find a degree sequence that you know will be a graph without actually drawing the graph?

2. Assume you have n vertices labeled 1, 2, 3, . . ., *n*. How many distinct (non-isomorphic) graphs can you make using these vertices?

3. Can two graphs with the same degree sequence be non-isomorphic? What is the fewest number of vertices in a graph such that two graphs have the same degree sequence but are not isomorphic?

4. If a certain graph is a tree and has average degree *a*, determine the number of vertices of the graph in terms of *a*.

5. If a certain graph has *n* vertices and average degree *a,* give a minimum value for *a* in terms of *n* that guarantees that the graph is connected.